Worldsheet Covariant Path Integral Quantization of Strings

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Abstract

We discuss a covariant functional integral approach to the quantization of the bosonic string. In contrast to approaches relying on non-covariant operator regularizations, interesting operators here are true tensor objects with classical transformation laws, even on target spaces where the theory has a Weyl anomaly. Since no implicit non-covariant gauge choices are involved in the definition of the operators, the anomaly is clearly separated from the issue of operator renormalization and can be understood in isolation, instead of infecting the latter as in other approaches. Our method is of wider applicability to covariant theories that are not Weyl invariant, but where covariant tensor operators are desired.

After constructing covariantly regularized vertex operators, we define a class of background-independent path integral measures suitable for string quantization. We show how gauge invariance of the path integral implies the usual physical state conditions in a very conceptually clean way. We then discuss the construction of the BRST action from first principles, obtaining some interesting caveats relating to its general covariance. In our approach, the expected BRST related anomalies are encoded somewhat differently from other approaches. We conclude with an unusual but amusing derivation of the value $D = 26$ of the critical dimension.

1 Introduction

We discuss the covariant functional integral quantization [1] of the bosonic string. Our approach is to take the functional integral seriously in the spirit of Fujikawa [2, 3], and to construct a covariantly regularized theory based on the methods introduced by Pauli and Villars [1, 4, 5, 6].

Our purpose is twofold. First, to show how familiar results in string theory can be recast in a covariant framework that that has conceptual and practical advantages over non-covariant approaches to operator regularization. And second, to illustrate in a simple setting some techniques for defining background independent functional integrals in generally covariant theories.

In our covariant approach, we find the same physical anomalies as in existing non-covariant approaches [8, 9, 10, 11, 12, 13]. However, the anomalies are manifested in the formalism in a different way that is often quite illuminating.

One of the main practical advantages of the covariant approach over the alternative operator approaches is that it permits one to construct interesting quantum operators that are finite, true covariant tensor objects, even in the presense of anomalies. There are no implicit gauge choices incorporated in our regularization, as in the non-covariant approaches. While this distinction is not that important for the critical string, it is of interest in the wider context of quantization of curved space-time theories that are not necessarily Weyl-invariant.

The covariant approach directly relates the anomalies to a failure of Weyl invariance at the quantum level. In contrast to the non-covariant approaches, this Weyl dependence is fully explicit in the action, and is not hidden in the measure or the regularization.

We show that anomalies are encoded in the fact that the trace of the energy momentum tensor (which is now a true tensor object) is not zero, but contributes contact terms when contracted with various operators.

In the familiar non-covariant regularizations, it is much more difficult to disentangle effects of the quantum gauge transformations from the implicit gauge choices that are made in the choice of regularization. In these approaches, operators defined in different coordinate frames are typically related not only by a coordinate transformation but also by a potentially anomalous gauge transformation. Disentangling the various contributions requires some gymnastics, and becomes conceptually quite involved. Indeed, it is often quite non-obvious whether the coordinate dependency introduced via a given operator regularization is in fact equivalent to a choice of gauge. If it is not, any quantum consistency conditions derived from them would be spurious and meaningless. The covariant approach of this paper does not suffer from this problem.

For these reasons, it is our hope that the covariant approach may be of some conceptual usefulness in the study of conformal field and string quantization. Our construction of background-independent path integral measures, and of the BRST action in the covariant approach, may have wider application in generally covariant theories.

The organization of the paper is as follows:

We first discuss the construction of covariantly regularized vertex operators. In contrast to existing approaches, these operators are true tensors satisfy classical Ward identities with respect to the energy-momentum tensor, which is a true covariant object.

After discussing the construction of background-independent path integral measures, we are ready to move on to string quantization. We show how gauge invariance of the path integral implies the usual physical state conditions in a way that is conceptually quite clean, if somewhat technically demanding. The correspondence with the Virasoro conditions is then demonstrated, and is rather indirect.

We then discuss the construction of the BRST action from first principles. Important in the current approach is the issue of background invariance of the BRST action, which we discuss rather carefully. We also discuss anomalies from the BRST point of view. In our approach, the BRST current is a true, non-anomalous tensor object, and the expected anomaly and physical state conditions are encoded differently from the operator approach. Their most natural expression in the formalism is in terms of the effective action and the antibracket.

For completeness, and because the calculation is sufficiently different from other approaches to make it interesting, we conclude by deriving the value $D = 26$ for the critical dimension.

This paper relies extensively on the techniques and results of [1].

2 Finite vertex operators and states

In this section we discuss a useful covariant renormalization of vertex operators in the covariant functional integral approach introduced in reference [1].

We will assume the form of the action introduced in that reference, which is the (X, χ) part of the full string action derived later and given by formula (34). Here the X are the matter fields and their Pauli-Villars partners are denoted by χ_i.

To illustrate how vertex operators may be renormalized in a coordinate-invariant way in the Pauli-Villars formalism, consider a path-integral calculation of the state corresponding to the operator

$$e^{ik\left(X+\sum_i \eta_i \chi_i\right)}.$$

Here the χ_i range over the Pauli-Villars regulator fields, and the η_i are complex numbers. Some of the χ_i are commuting real scalars, and some are complex Grassmann scalars [1]. We require

$$\eta_i = 0, \quad \text{for } \chi_i \text{ Grassmann.}$$

We shall show that this expression will provide a full renormalization of the undressed insertion e^{ikX} once the masses of the regulator fields and the constants η_i have been chosen to make path integrals containing it finite.

To obtain the state corresponding to this vertex operator, we need to calculate the path integral

$$\begin{aligned}\int_{(X,\chi)_{\partial D}=(X_b,\chi_b)} & [dX] \wedge [d\bar{\chi}] \wedge [d\chi] \\ & \times \exp\biggl(-\frac{1}{2} \int d^2x \left\{ 4\,\partial X \bar{\partial} X + m^2 X^2 + \sum_i \left(4\,\partial \bar{\chi}_i \bar{\partial} \chi_i + M_i^2 \chi_i^2 \right) \right\} \\ & \qquad + ik \left\{ X(0) + \sum_i \eta_i \chi_i(0) \right\} \biggr)\end{aligned}$$

where the path integral is taken over configurations of the fields on the unit disc D with boundary ∂D to obtain a functional of the boundary configuration (X_b, χ_b) of the fields.[1] At this point we ignore the ghost contribution, which will be discussed later.

In [1], a careful definition of the path integral measure was provided. It was shown there that the full measure, after including the Pauli-Villars contributions, is invariant under arbitrary variations of the metric, and also separately under arbitrary diffeomorphisms acting only on the fields. Putting

[1] Our normalization of the action corresponds to replacing $\alpha' \to \frac{1}{2\pi}$.

these two facts together, it follows that the measure is invariant under full diffeomorphisms and Weyl transformations, a desirable property for a conformal field theory. Note that this measure differs from the usual Fujikawa measure for X only, which is not conformally invariant.

Writing $X = X_c + \bar{X}$ where X_c is the extremum of the term in the exponent with $X_c(\partial D) = X_b$, and $\bar{X}(\partial D) = 0$, the path integral over $\bar{X}$ is Gaussian and contributes an overall normalization, and we obtain the matter contribution

$$Z\, e^{-S(X_c)+ik\, X_c(0)} \tag{1}$$

The stationary configuration X_c is obtained by varying X in the bulk of D to get

$$-4\,\partial\bar{\partial}X_c + m^2 X_c - ik\,\delta^2(x) = 0.$$

This is solved by

$$X_c = X_0 + \sum_{n>0}(z^n X_n + \bar{z}^n X_{-n}) - \frac{ik}{4\pi}\ln m^2 z\bar{z}$$

in the limit as $m \to 0$, remembering that $K_0(mr) \to -\ln mr$ as $m \to 0$, or simply noting that $\partial\bar{\partial}\ln z\bar{z} = \pi\delta^2(x)$.

Note that the extremum X_c is complex, so that the above decomposition $X = X_c + \bar{X}$, with $\bar{X}$ real, describes an integration path in field space that has been displaced away from the real line. It is straightforward to convince oneself that, as in the one-dimensional analogue

$$\int_{-\infty}^{\infty} dz\, e^{-az^2} = \int_{-\infty+ib}^{\infty+ib} dz\, e^{-az^2},$$

such a displacement does not change the value of the integral, since no poles are crossed and the Gaussian decays sufficiently rapidly.

Note, however, that the boundary values $X_c(|z|=1) = X_b(z)$ of the field are constrained to be real, which implies that

$$X_0 = \frac{ik}{4\pi}\ln m^2 + x_0,$$

where x_0 is real, and we may write

$$X_c = x_0 + \sum_{n>0}(z^n X_n + \bar{z}^n X_{-n}) - \frac{ik}{4\pi}\ln z\bar{z}$$

Inserting the stationary solution, the path integral becomes

$$Z\, e^{ikx_0} \exp\left\{-2\pi \sum_{n>0} n X_n X_{-n}\right\} \exp\left\{-\frac{1}{2}\,\frac{(ik)^2}{4\pi}\,\ln 0^2\right\}.$$

The first two exponentials are just proportional to [12]

$$e^{ikx_0}\,|0\rangle_X\,.$$

The term indicated by $\ln 0^2$ in the last exponential would be divergent in the absence of the Pauli-Villars contributions. However, it is neatly canceled once we include the latter. For the Pauli-Villars fields, the stationary solutions may be approximated for large M_i by Bessel functions as

$$\chi_i^c \to \chi_0^i + \sum_{n>0}\left(z^n \chi_n^i + \bar{z}^n \chi_{-n}^i\right) + \frac{ik}{4\pi}\, 2\eta_i\, K_0(M_i r).$$

In the infinite mass limit, the Bessel functions tend to zero on the boundary. As a result, we get no $\exp ik\chi_0^i$ contributions, in contrast with the field X above. The path integral becomes

$$Z' \prod_i \exp\left\{-2\pi \sum_{n>0} n\chi_n^i \chi_{-n}^i\right\} \exp\left\{-\frac{1}{2}\,\frac{(ik)^2}{4\pi}\left(-\sum_i 2\,\eta_i^2\, K_0(M_i 0)\right)\right\}.$$

The first exponent is proportional to the vacuum $|0\rangle_{PV}$ of the Pauli-Villars fields. The potentially divergent terms in the exponent contribute

$$\begin{aligned}
&\lim_{r\to 0}\left(\ln r^2 - \sum_i 2\,\eta_i^2\, K_0(M_i r)\right)\\
&\qquad = \lim_{r\to 0}\left(\ln r^2 + \sum_i \eta_i^2\, \ln M_i^2 r^2\right)\\
&\qquad = \sum_i \eta_i^2\, \ln M_i^2 + \lim_{r\to 0}\left(1 + \sum_i \eta_i^2\right)\ln r^2,
\end{aligned}$$

remembering that $K_0(Mr) \to -\ln Mr$ as $r \to 0$. Although we have arbitrarily introduced an short-distance cutoff r, the Pauli-Villars result is independent of the precise cutoff method used once we take the continuum limit.

It is indeed this property of the Pauli-Villars regularization that makes it suitable as a non-perturbative, coordinate-invariant regularization.

We now impose the conditions

$$0 = 1 + \sum_i \eta_i^2, \tag{2}$$

$$0 = \sum_i \eta_i^2 \ln \frac{M_i^2}{\mu^2}, \tag{3}$$

on the coefficients η_i and the Pauli-Villars masses M_i. Here μ is an arbitrary *finite* renormalization scale. These conditions can always be satisfied while taking $M_i \to \infty$ as long as there are enough Pauli-Villars fields. Since the parameter μ constrains the way we take the $M_i \to \infty$ limit, the resulting path integral measure depends implicitly on μ. In fact, we are really defining a one-parameter family of path integral measures depending on μ.

The above expression is then finite

$$\sum_i \eta_i^2 \ln M_i^2 = - \ln \mu^2,$$

and the final result of the path integral is proportional to

$$\mu^{-k^2/4\pi} \, e^{ikx_0} \, |0\rangle_X \otimes |0\rangle_{PV} \equiv |k\rangle$$

Although the μ-dependent prefactor is perfectly finite, physical states should not depend on the renormalization scale. We can indeed compensate the prefactor by instead considering the finitely renormalized insertion

$$\mu^{k^2/4\pi} e^{ik\left(X + \sum_i \eta_i \chi_i\right)} \tag{4}$$

whose correlation functions will be independent of μ.

3 Two-point functions

Let us calculate the two-point function

$$\left\langle e^{ik_1\left(X(w) + \sum_i \eta_i \chi_i(w)\right)} \, e^{ik_2\left(X(0) + \sum_i \eta_i \chi_i(0)\right)} \right\rangle$$

via a path integral. The presence of the Pauli-Villars terms in the exponents will make the result finite without any additional renormalization.

We proceed as in the previous section. The stationary solutions now have two sources and the boundary conditions are different from those in the previous section. On the plane with vanishing boundary conditions on X and χ_i at infinity, the solutions are

$$X_c = -\frac{ik}{4\pi}\,\ln m^2 z\bar{z} - \frac{ik}{4\pi}\,\ln m^2 (z-w)\overline{(z-w)}$$

and

$$\chi_i^c = \frac{ik}{4\pi}\,2\eta_i\,K_0(M_i|z|) + \frac{ik}{4\pi}\,2\eta_i\,K_0(M_i|z-w|).$$

Similar to (1), the result of the path integration is

$$Z\,e^{-S(X_c)+ik_1\,X_c(w)+ik_2\,X_c(0)}$$

for the matter contribution, and similarly for the Pauli-Villars fields. After inserting the stationary solutions into this expression, we obtain

$$\begin{aligned} Z\exp\Bigg\{-\frac{1}{2}\,\frac{1}{4\pi}\Bigg[&(ik_1)^2\lim_{r\to 0}\Bigg(\ln m^2 r^2 - \sum_i 2\,\bar{\eta}_i^2\,K_0(M_i r)\Bigg)\\ &+2\,(ik_1)(ik_2)\Bigg(\ln m^2 w\bar{w} - \sum_i 2\,\bar{\eta}_i^2\,K_0(M_i|w|)\Bigg)\\ &+(ik_2)^2\lim_{r\to 0}\Bigg(\ln m^2 r^2 - \sum_i 2\,\bar{\eta}_i^2\,K_0(M_i r)\Bigg)\Bigg]\Bigg\}. \end{aligned}$$

Assuming $w \neq 0$, we now use

$$\lim_{M_i\to\infty} K_0(M_i|w|) = 0.$$

The total coefficient of $\ln r$ is

$$1 + \sum_i \eta_i^2 = 0,$$

by the Pauli-Villars condition (2). We obtain

$$\begin{aligned} &Z\exp\left\{\frac{1}{2}\,\frac{1}{4\pi}\left((k_1+k_2)^2\ln m^2 + 2\,k_1k_2\ln\bar{w}w + (k_1^2+k_2^2)\sum_i \eta_i^2\ln M_i^2\right)\right\}\\ &\quad = Z\,m^{(k_1+k_2)^2/4\pi}\,\mu^{-k_1^2/4\pi}\mu^{-k_2^2/4\pi}\,|w|^{k_1k_2/2\pi}, \end{aligned}$$

where we have used the Pauli-Villars condition (3).

We see that, as $m \to 0$, the correlation function vanishes unless $k_1 + k_2 = 0$. The final result is finite, and is given by

$$\begin{aligned} &\left\langle e^{ik_1\left(X(w)+\sum_i \eta_i\chi_i(w)\right)}\, e^{ik_2\left(X(0)+\sum_i \eta_i\chi_i(0)\right)} \right\rangle \\ &\qquad = \langle 1 \rangle \begin{cases} 0 & \text{if } 0 \neq k_1 + k_2 \\ \mu^{-k_1^2/4\pi}\, \mu^{-k_2^2/4\pi}\, |w|^{k_1 k_2/2\pi} & \text{if } 0 = k_1 + k_2 \end{cases} \end{aligned}$$

As discussed in the previous section, the μ-dependent prefactors can be trivially compensated by a finite renormalization of the vertices, in which case the result becomes independent of μ.

So far we have worked on the plane. We may add the point at infinity, in which case the only modification to the above analysis is the existence of a zero mode $\bar{X}_0$ over which we should integrate in the path integral. This would contribute an additional factor

$$\int d\bar{X}_0\, e^{ik_1\bar{X}_0 + ik_2\bar{X}_0} = 2\pi\, \delta(k_1 + k_2),$$

to the above result.

4 General vertex operators

We now consider the construction of higher vertex operators in the covariant approach. These are obtained by multiplying derivatives of the fields with the tachyon. For example, the matter part of the graviton is of the form

$$\partial X\, \bar{\partial} X\, e^{ikX},$$

where we have suppressed target space indices. In the path integral, these bare insertions are not finite due to self-contractions. In the covariant Pauli-Villars approach, we obtain a finite insertion by considering instead

$$\partial \left(X + \sum_i \eta_i \chi_i \right) \bar{\partial} \left(X + \sum_i \eta_i \chi_i \right)\; e^{ik\left(X + \sum_i \eta_i \chi_i\right)},$$

In addition to self-contractions already present in the exponential, which were shown to be finite in the previous sections, this insertion has derivatives of self-contractions of the form

$$\left\langle \partial \left(X + \sum_i \eta_i \chi_i \right) \left(X + \sum_i \eta_i \chi_i \right) \right\rangle ,$$
$$\left\langle \bar{\partial} \left(X + \sum_i \eta_i \chi_i \right) \left(X + \sum_i \eta_i \chi_i \right) \right\rangle ,$$
$$\left\langle \partial \left(X + \sum_i \eta_i \chi_i \right) \bar{\partial} \left(X + \sum_i \eta_i \chi_i \right) \right\rangle . \tag{5}$$

But these are finite. Indeed, we have for small z,

$$\begin{aligned}
&\left\langle \left(X + \sum_i \eta_i \chi_i \right)_z \left(X + \sum_i \eta_i \chi_i \right)_0 \right\rangle \\
&\to -\frac{1}{4\pi} \left(\ln m^2 \bar{z} z + \sum_i \eta_i^2 \ln M_i^2 \bar{z} z \right) \\
&= \left(\ln m^2 + \sum_i \eta_i^2 \ln M_i^2 \right) + \left(1 + \sum_i \eta_i^2 \right) \ln \bar{z} z \\
&= \ln \frac{m^2}{\mu^2} + 0,
\end{aligned}$$

becoming independent of z as $z \to 0$ by the two previously introduced conditions on η_i and M_i. This diverges as $m \to 0$, but taking a derivative, we see that the desired contractions (5) are finite, indeed zero, independent of m, and the result follows.

This is easily generalized, so that the replacement

$$X \to X + \sum_i \eta_i \chi_i$$

is sufficient to render the entire tower of vertex insertions finite.

5 Covariant Ward identities

In the previous sections, we saw that the full vertex insertion, including the Pauli-Villars contributions, is finite under suitable conditions on the Pauli-Villars masses. We will now show that the tachyon vertex satisfies the classical *scalar* Ward identities with respect to coordinate transformations. We will then relate these to the more familiar anomalous Ward identities found in the non-covariant operator formalism.

The change of variables formula for the path integral under a deformation $X \to X^\lambda = X \circ f_{-\lambda}$ of the dynamic fields and their Pauli-Villars partners along a vector field v generating the flow $x \to f_\lambda(x)$, where λ is a real parameter along the flow, can be written as [1]

$$\int [dX]^\lambda_{PV}\, V^\lambda(x)\, e^{-S(X^\lambda,\bar{\chi}^\lambda,\chi^\lambda)} = \int [dX]_{PV}\, V(x)\, e^{-S(X,\bar{\chi},\chi)}, \tag{6}$$

where

$$[dX]_{PV} \equiv [dX] \wedge [d\bar{\chi}] \wedge [d\chi],$$

and $V^\lambda \equiv V(X^\lambda, \bar{\chi}^\lambda, \chi^\lambda)$. It should be emphasized that we are not deforming the metric, so that the above transformation would be a classical symmetry only if v is conformal. However, the Ward identities below will be valid for arbitrary deformations v satisfying suitable boundary conditions. For example, on the plane, derivatives of v should vanish sufficiently fast at infinity, which excludes globally conformal v.

The Ward identities are obtained from this formula by differentiating with respect to λ. As shown in [1], the full measure $[dX]^\lambda_{PV}$, due to the inclusion of the Pauli-Villars fields, is invariant under the deformation $(X, \bar{\chi}, \chi) \to (X^\lambda, \bar{\chi}^\lambda, \chi^\lambda)$, and so independent of λ. We therefore obtain, after differentiation,

$$\left\langle \frac{d}{d\lambda}\, V^\lambda(x) \cdots \right\rangle + \frac{1}{4\pi} \int d^2y\, \sqrt{g}\, h^{kl}(y)\, \left\langle T^\lambda_{kl}(y)\, V^\lambda(x) \cdots \right\rangle = 0, \tag{7}$$

where T^{kl} is the full energy-momentum tensor including the Pauli-Villars contribution,

$$\begin{aligned} \frac{d}{d\lambda}\, V^\lambda(x) &= -\mathcal{L}_v V^\lambda(x) + \frac{\delta V^\lambda}{g^{ij}}\, \mathcal{L}_v g^{ij}(x) \\ &= -\mathcal{L}_v V^\lambda(x) + \frac{\delta V^\lambda}{g^{ij}}\, h^{ij}(x) \end{aligned} \tag{8}$$

where $\mathcal{L}_v$ denotes the Lie derivative, and

$$h^{kl} \equiv -\nabla^k v^l - \nabla^l v^k.$$

Since the metric is not being varied under v, operators depending on the metric need the second term on the right hand side of (8) to compensate contributions proportional to $-\mathcal{L}_v g_{ij}$.

It is important to note that the transformation $-\mathcal{L}_v V$ for V appearing in the Ward identity is the *classical* one, despite the fact that V is a fully renormalized, finite insertion. This is not what one might naively expect from previous acquaintance with the operator formalism, where the Ward identity for V has an anomalous term, corresponding to the anomalous dimension of the operator.

However, further thought shows that there is no contradiction. This formula is in fact correct in the full quantum theory, and is consistent with the operator formalism. To understand this, let us specialize to the plane, where $dw\, d\bar{w}\sqrt{g} = d^2 w$, and consider the example

$$V \equiv e^{ik\left(X + \sum_i \eta_i \chi_i\right)},$$

for which the $\frac{\delta V}{\delta g^{ij}}$ term vanishes. The above then becomes

$$\begin{aligned}
\left\langle \left(v^z \partial + v^{\bar{z}} \bar{\partial}\right) V_z \cdots \right\rangle = &- \frac{1}{\pi} \int d^2 w\, \bar{\partial} v^w \ \langle T_{ww}\, V_z \cdots \rangle \\
&- \frac{1}{\pi} \int d^2 w \left(\partial v^w + \bar{\partial} v^{\bar{w}}\right) \langle T_{w\bar{w}}\, V_z \cdots \rangle \\
&- \frac{1}{\pi} \int d^2 w\, \partial v^{\bar{w}} \ \langle T_{\bar{w}\bar{w}}\, V_z \cdots \rangle .
\end{aligned} \tag{9}$$

We shall assume growth conditions on v so that the integrals on the right hand side exist and may be partially integrated without surface contributions. For example, on the plane, v may at most tend to a constant vector field at infinity. When we add the point at infinity to obtain the sphere, v must be everywhere defined, and includes the fields $v^z = a + bz + cz^2$ generating the group $PSL(2, \mathbf{C})$ of unimodular Möbius transformations. However, we emphasize that this formula is valid for general v, not just holomorphic or conformal v.

The consistency of the above result with the usual operator formalism may be understood by noticing that, although classically $T_{w\bar{w}} = 0$, in a covariantly regularized quantum field theory $T_{w\bar{w}}$ can lead to contact terms in expectation values [1, 15, 16, 17, 18]. Such contact terms were obtained from axiomatic considerations in [15, 16, 17] and were discussed and calculated in much detail in [1] using a covariant Pauli-Villars regularization. A similar calculation from first principles will be done below, but before we do that, we illustrate a simpler, though indirect, derivation of the contact terms.

This proceeds by noting that the Pauli-Villars-regularized objects are by construction coordinate-invariant and finite. The above derivation of the Ward identity is therefore rigorous, and may be used as a starting point for inferring the contact terms. Taylor-expanding the exponential and performing single and double contractions with T_{zz}, we obtain, in the limit of infinite Pauli-Villars mass, [12]

$$\begin{aligned}\left\langle T_{ww}\, e^{ik(X+\sum\eta_i\chi_i)(z)}\cdots\right\rangle = {} & \frac{\alpha' k^2/4}{(w-z)^2}\left\langle e^{ik(X+\sum\eta_i\chi_i)(z)}\cdots\right\rangle \\ & + \frac{1}{w-z}\,\partial_z\left\langle e^{ik(X+\sum\eta_i\chi_i)(z)}\cdots\right\rangle + \cdots .\end{aligned}$$

and similarly for the product with $T_{\bar{w}\bar{w}}$. The final ellipsis stands for any terms that do not arise from self-contractions among fields in $T_{ww}\, e^{ikX_z}$, and may also contain terms proportional to $\chi_i(z)$ whose matrix elements will vanish, in the limit of infinite Pauli-Villars mass, as long as no additional insertions are at z. For their explicit form, see the calculation later in this section. As above, T_{ww} denotes the full energy-momentum tensor including the contributions of the Pauli-Villars auxiliary fields.

The above calculation is familiar from the operator formalism, but deserves a few comments in the present context. First, note that the insertions are already regularized, since divergences due to contractions of fields at the same point are cancelled by the contributions of the Pauli-Villars fields as in the previous section and reference [1]. It is important, though, to make sure that we are not overlooking contact terms due to contractions of Pauli-Villars terms in T_{ww} with e^{ikX_z}. This is easily verified for single contractions, where the relevant terms would be proportional to $\langle \partial_w\chi\, \bar{\chi}\rangle \sim \partial_w K_0(Mr)$. Since the Bessel function $K_0(Mr)$ is positive and has area $2\pi/M^2$ in two dimensions, it tends to the zero distribution as we take $M \to \infty$, and therefore so does its derivative $\partial_w K_0(Mr)$. Slightly less obvious are the contributions due to double contractions. A typical term is proportional to $\langle(\partial_w\chi)^2\,\chi^2\rangle$, whose Fourier transform can be calculated as in [1] to be proportional to

$$\frac{1}{M^2}\int_{2M}^{\infty} d\mu\, c(\mu, M)\,\frac{\mu^2\,(p_1 - ip_2)^2}{p^2+\mu^2},$$

where $c(\mu, M)$ is a (spectral) function of unit area with support on $[2M, \infty)$. Due to the lower bound on the integration and the unit area property, this indeed becomes $(p_1 - ip_2)^2/M^2 \to 0$ in the limit as $M \to \infty$.

We may now obtain the contact terms by inserting the above result into the Ward identity (9), performing partial integrations,[2] and using

$$\pi\, \delta^2(w-z) = \partial_w \partial_{\bar{w}} \, \ln |w-z|^2,$$

we find

$$\frac{1}{\pi} \int d^2 w \, (\partial v^w + \bar{\partial} v^{\bar{w}}) \, \langle T_{w\bar{w}} \, V_z \cdots \rangle = \frac{\alpha' k^2}{4} \, (\partial v^z + \bar{\partial} v^{\bar{z}}) \, \langle V_z \cdots \rangle + \cdots .$$

Since this is true for general v, we obtain

$$T_{w\bar{w}} \, V_z = \pi \, \frac{\alpha' k^2}{4} \, \delta^2(w-z) \, V_z + \cdots . \tag{10}$$

Since this formula is so important in what follows, and since it is not usually calculated in more familiar regularization schemes, we now verify it by a direct calculation. We have

$$\begin{aligned}
&T_{w\bar{w}} \, e^{ik\left(X(z) + \sum_i \eta_i \chi_i\right)} \\
&\quad = \frac{\pi}{2} \left(m^2 \phi^2 + \sum_i M_i^2 \chi_i^2 \right) \, e^{ik\left(X(z) + \sum_i \eta_i \chi_i(z)\right)} \\
&\quad = \frac{\pi}{2} \Bigg\{ \frac{(ik)^2}{2!} \left(m^2 \left\langle X^2(w) \, X^2(z) \right\rangle + \sum_i \eta_i^2 M_i^2 \left\langle \chi_i^2(w) \, \chi_i^2(z) \right\rangle \right) \\
&\qquad\quad + ik \left(m^2 \, X(w) \, \langle X(w) \, X(z) \rangle + \sum_i \eta_i M_i^2 \, \chi_i(w) \, \langle \chi_i(w) \, \chi_i(z) \rangle \right) \Bigg\} \times \\
&\qquad \times e^{ik\left(X(z) + \sum_i \eta_i \chi_i(z)\right)} + \cdots ,
\end{aligned} \tag{11}$$

where the ellipsis denotes the non-contracted remainder. The double contractions in the first line were already partially calculated in [1]. The result, writing $w \equiv x_1 + ix_2$ and $z \equiv y_1 + iy_2$, is

$$\frac{1}{(\pi/2)} \, \frac{(ik)^2}{2!} \int \frac{d^2 p}{(2\pi)^2} \, e^{-ip\cdot(x-y)} \, \frac{1}{16} \cdot \frac{\pi}{3} \left(\int_{2m}^{\infty} d\mu \, \frac{c(\mu, m)}{m^2} \, \frac{\mu^4}{p^2 + \mu^2} + PV \right), \tag{12}$$

[2]The surface terms arising from partial integrations will be zero if v goes to a constant at infinity. Also note that as $w \to z$, the distributions $1/(w-z)^n$ are integrable and do not spoil our ability to perform partial integrations.

where the unit area spectral function is explicitly given by

$$c(\mu, m) \equiv \frac{24\, m^4}{\mu^5 \sqrt{1 - 4m^2/\mu^2}}\, \theta(\mu - 2m).$$

Writing

$$\frac{\mu^4}{p^2 + \mu^2} = \mu^2 - \frac{\mu^2 p^2}{p^2 + \mu^2}, \tag{13}$$

we obtain a term

$$\int_{2m}^{\infty} d\mu\, \frac{c(\mu, m)}{m^2}\, \mu^2 + \sum_i \eta_i^2 \int_{2M_i}^{\infty} d\mu\, \frac{c(\mu, M_i)}{M_i^2}\, \mu^2.$$

Changing variables from μ to $\nu \equiv 2m\mu$ and $\nu \equiv 2M_i\mu$ respectively removes all mass-dependence from this formula, and the result is proportional to

$$1 + \sum_i \eta_i^2 = 0.$$

The second term in the identity (13) contributes

$$\begin{aligned}
&\int_{2m}^{\infty} d\mu\, \frac{c(\mu, m)}{m^2}\, \frac{\mu^2 p^2}{p^2 + \mu^2} + \sum_i \eta_i^2 \int_{2M_i}^{\infty} d\mu\, \frac{c(\mu, M_i)}{M_i^2}\, \frac{\mu^2 p^2}{p^2 + \mu^2} \\
&\quad = 4 \cdot \frac{3}{2} \int_1^{\infty} \frac{d\nu}{\nu^4}\, \frac{1}{\sqrt{\nu^2 - 1}} \left(\frac{\nu^2 p^2}{p^2 + 4m^2\nu^2} + \sum_i \eta_i^2 \frac{\nu^2 p^2}{p^2 + 4M_i^2 \nu^2} \right) \\
&\quad \to 4 \cdot \frac{3}{2} \int_1^{\infty} \frac{d\nu}{\nu^4}\, \frac{\nu^2}{\sqrt{\nu^2 - 1}} \\
&\quad = 4 \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot B\big(\tfrac{1}{2}, 1\big) \\
&\quad = 4 \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{\Gamma\big(\frac{1}{2}\big)\, \Gamma\big(1\big)}{\Gamma\big(\frac{3}{2}\big)} \\
&\quad = 6,
\end{aligned}$$

where the limit $M_i \to \infty$ and $m \to 0$ has been taken in the third line, which makes the Pauli-Villars contributions vanish.

The contribution (12) therefore becomes

$$-\frac{1}{(\pi/2)}\, \frac{(ik)^2}{2!} \int \frac{d^2p}{(2\pi)^2}\, e^{-ip\cdot(x-y)}\, \frac{1}{16} \cdot \frac{\pi}{3} \cdot 6 = \frac{k^2}{8}\, \delta^2(w - z).$$

This will indeed give the contact term (10) when inserted in (11).

We still need to calculate the contributions of the single contractions appearing in (11). We have

$$
\begin{aligned}
&ik\biggl(m^2\,X(w)\,\langle X(w)\,X(z)\rangle + \sum_i \eta_i M_i^2\,\chi_i(w)\,\langle \chi_i(w)\,\chi_i(z)\rangle\biggr) \\
&\quad = ik\biggl(X(w)\,m^2\,\frac{1}{2\pi}\,K_0(m|w-z|) + \sum_i \eta_i\,\chi_i(w)\,M_i^2\,\frac{1}{2\pi}\,K_0(M_i|w-z|)\biggr) \\
&\quad \to ik\,\delta^2(w-z)\sum_i \eta_i\,\chi_i(w)
\end{aligned}
$$

in the limit $m \to 0$ and $M_i \to \infty$. The full expression is thus

$$
\begin{aligned}
T_{w\bar{w}}\,e^{ik\left(X(z)+\sum_i \eta_i\chi_i\right)} = \delta^2(w-z)\left\{\frac{k^2}{8} + ik\sum_i \eta_i\,\chi_i(z)\right\} e^{ik\left(X(z)+\sum_i \eta_i\chi_i(z)\right)} \\
+\cdots, \qquad (14)
\end{aligned}
$$

which differs from (11) by the presence of additional contributions proportional to the Pauli-Villars fields. However, as long as no additional insertions are at z, the matrix elements of these contributions go to zero when $M_i \to \infty$, since in this limit the range of the propagator goes to zero and χ_i becomes non-dynamical. Modulo this condition, we therefore find

$$
T_{w\bar{w}}\,e^{ik\left(X(z)+\sum_i \eta_i\chi_i\right)} = \frac{k^2}{8}\,\delta^2(w-z)\,e^{ik\left(X(z)+\sum_i \eta_i\chi_i(z)\right)} + \cdots. \qquad (15)
$$

To see that our analysis is consistent with the usual operator formalism result, we restate our Ward identity (9) as follows

$$
\begin{aligned}
&-\frac{1}{\pi}\int d^2w\,\bar{\partial}v^w\ \langle T_{ww}\,V_z\cdots\rangle - \frac{1}{\pi}\int d^2w\,\partial v^{\bar{w}}\ \langle T_{\bar{w}\bar{w}}\,V_z\cdots\rangle \\
&\qquad = \bigl\langle (v^z\partial + v^{\bar{z}}\bar{\partial})\,V_z\cdots\bigr\rangle + \frac{1}{\pi}\int d^2w\,(\partial v^w + \bar{\partial}v^{\bar{w}})\ \langle T_{w\bar{w}}\,V_z\cdots\rangle \\
&\qquad = \bigl\langle (v^z\partial + v^{\bar{z}}\bar{\partial})\,V_z\cdots\bigr\rangle + \frac{\alpha' k^2}{4}\,(\partial_i v^i)\ \langle V_z\cdots\rangle + \cdots. \qquad (16)
\end{aligned}
$$

where we used the explicit result (15) for the contact term. This is the familiar anomalous identity from the operator formalism.

It is now clear exactly how the coordinate dependence arises in the operator formalism. In [1] it was shown that the full renormalized energy-momentum $T_{ij}dx^i \otimes dx^j$, including Pauli-Villars contributions, is a coordinate-invariant, true tensor quantity. But its component $T_{w\bar{w}}$ of course depends on the coordinate system, so that by moving the $T_{w\bar{w}}$ contribution to the right hand side in (16), one is explicitly making both sides of the equation coordinate-dependent.

In the current formalism, the anomalous dimension of the insertion is encoded in the contact contraction (15). Since $T_{z\bar{z}}$ is precisely the generator of dilations, we have directly related the anomalous dimension to scale-dependence, a relationship that is somewhat obscured in the usual operator treatment. We also note that the operator formalism result is reproduced after ignoring the nonsingular terms, represented by the dots, in (15). This may be done for the Ward identity but needs care in general amplitudes, where these terms may contribute.

From (16), the precise relationship between our Pauli-Villars-regulated vertex V and the operator formalism vertex operator $\hat{V}$ is now clear. Define

$$\hat{V} = V$$

on the plane with trivial metric, and deform $\hat{V}$ according to the transformation law

$$\delta_v \hat{V} \equiv v^i \partial_i \hat{V} + \frac{\alpha' k^2}{4} \left(\partial_i v^i\right) \hat{V}$$

as we deform only the metric along the flow of a vector field v holomorphic in a neighbourhood of the insertion. Then $\hat{V}$ coincides with the operator formalism vertex.

The anomalous term in the transformation $\delta_v \hat{V}$ is an artifact of the coordinate-dependent definition of $\hat{V}$, and is somewhat unnatural in the present formalism. On the other hand, the covariantly regularized V transforms as a scalar, without this extra term, but requires the nonzero contraction with $T_{z\bar{z}}$ in the Ward identity.

It should be stressed that both the coordinate-independent V and the coordinate-dependent $\hat{V}$ are *finite*, renormalized insertions. The difference between them is a finite, but coordinate-dependent quantity.

What about conformal transformations? Notice that (9) is true for arbitrary deformations v that goes to a constant at infinity. Now consider a vector field v that is holomorphic

$$\partial_{\bar{z}} v^z = 0 = \partial_z v^{\bar{z}}$$

on a disc-shaped neighborhood D of z. Since classically such a deformation would be a symmetry, one would expect the transformation $\mathcal{L}_v V$ to be generated by conserved charges. Indeed, as shown in [1], away from additional insertions the conservation law

$$\partial_{\bar{w}} T_{ww} + \partial_w T_{\bar{w}w} = 0 = \partial_w T_{\bar{w}\bar{w}} + \partial_{\bar{w}} T_{ww}$$

holds, so we find, denoting by $\bar{D}$ the complement of D,

$$\begin{aligned}
\left\langle (v^z \partial + v^{\bar{z}} \bar{\partial})\, V_z \cdots \right\rangle = & -\frac{1}{\pi} \int_{\bar{D}} d^2 w \,\left\langle \bar{\partial}\, (v^w\, T_{ww} + v^{\bar{w}}\, T_{\bar{w}w})\, V_z \cdots \right\rangle \\
& -\frac{1}{\pi} \int_{\bar{D}} d^2 w \,\langle \partial\, (v^w\, T_{w\bar{w}} + v^{\bar{w}}\, T_{\bar{w}\bar{w}})\, V_z \cdots \rangle \\
& -\frac{1}{\pi} \int_{D} d^2 w\, (\partial v^w + \bar{\partial} v^{\bar{w}})\, \langle T_{w\bar{w}}\, V_z \cdots \rangle
\end{aligned}$$

In the first two terms, since w is outside D, the contact term in the contraction $T_{\bar{w}w} V_z$ does not contribute. We obtain

$$\begin{aligned}
\left\langle (v^z \partial + v^{\bar{z}} \bar{\partial})\, V_z \cdots \right\rangle = & \frac{i}{\pi} \oint_{\partial D} dw\, \langle v^w\, T_{ww}\, V_z \cdots \rangle - \frac{i}{\pi} \oint_{\partial D} d\bar{w}\, \langle v^{\bar{w}}\, T_{\bar{w}\bar{w}}\, V_z \cdots \rangle \\
& -\frac{1}{\pi} \int_{D} d^2 w\, (\partial v^w + \bar{\partial} v^{\bar{w}})\, \langle T_{w\bar{w}}\, V_z \cdots \rangle
\end{aligned}$$

The first line represents the contractions with the usual Virasoro charges, while the second line is an anomalous term due to quantum breaking of scale invariance. As shown in equation (16), this term is exactly the anomalous transformation of the operator-formalism vertex. In the present covariant formalism, it is more natural, and perhaps less confusing, to say that since conformal invariance is broken, the conserved charges do *not* generate the transformation, with the needed correction given by the contact term in the second line.

6 Ghost path integral

To make this article more self-contained, we give a short overview of the ghost component of the path integral, which was not considered in the previous sections. For the purposes of this calculation, we will not need the Pauli-Villars partners for the ghosts. However, the full action does include such additional fields, and will be derived in section 10.

As in the previous section, the ghost path integral can be performed by expanding around a stationary solution, which can be written on the disc as

$$b_{zz} = \sum_{m=-\infty}^{-2} b_m z^{-2-m}, \tag{17}$$

$$c^z = \sum_{m=-\infty}^{1} c_m z^{1-m}, \tag{18}$$

and similarly for $z \leftrightarrow \bar{z}$. The state on the boundary can therefore be written as a wave function

$$\psi(\cdots, b_{-4}, b_{-3}, b_{-2}, c_1, c_0, c_{-1}, \cdots) \equiv \langle \cdots, b_{-4}, b_{-3}, b_{-2}, c_1, c_0, c_{-1}, \cdots | \psi\rangle$$

on the configuration space parameterized by the above expansion coefficients. This wave function is given by the path integral as [14]

$$\begin{aligned} \psi &= \mathcal{Z}\, e^{-\int_D (b_{zz}\partial_{\bar{z}} c^z + c^z \partial_{\bar{z}} b_{zz}) + \int_{\delta D} dz\, b_{zz} c^z + (z\leftrightarrow\bar{z})} \\ &= \mathcal{Z}\, e^0 \\ &= \mathcal{Z}, \end{aligned}$$

where the fields b and c in the exponent denote the stationary solution. The boundary term in the action ensures that the equations of motion are satisfied in the presence of the boundary. Ignoring the antiholomorphic contribution for now, we may write this state in terms of the conventionally defined ghost ground state, given by

$$\langle \cdots, b_{-2}, b_{-1}, c_0, c_{-1}, \cdots | \downarrow\rangle = 1,$$

by calculating

$$\begin{aligned} \langle \cdots, b_{-2}, b_{-1}, c_0, c_{-1}, \cdots | \psi\rangle &= \int dc_1 \; \langle b_{-1} | c_1\rangle \; \langle \cdots, b_{-4}, b_{-3}, b_{-2}, c_1, c_0, c_{-1}, \cdots | \psi\rangle \\ &= \int dc_1\, e^{i b_{-1} c_1}\, \mathcal{Z} \\ &= \mathcal{Z}\, b_{-1} \\ &= \mathcal{Z}\; \langle \cdots, b_{-2}, b_{-1}, c_0, c_{-1}, \cdots | \, b_{-1} \, | \downarrow\rangle \,, \end{aligned}$$

where we have used the fact that b_{-n} and c_n are conjugate variables. We find

$$|\psi\rangle = \mathcal{Z}\, b_{-1}\, |{\downarrow}\rangle\,,$$

or, including the antiholomorphic contribution,

$$|\psi\rangle = \mathcal{Z}\, b_{-1}\bar{b}_{-1}\, |{\downarrow}\rangle\,.$$

Let us also calculate the path integral over the complement of the disc on the sphere. Taking into account the transformations

$$\partial_z \to -z^2\,\partial_z, \tag{19}$$

$$dz\otimes dz \to \frac{1}{z^4}\,dz\otimes dz, \tag{20}$$

under $z\to 1/z$, the stationary solutions that are regular at infinity can be written as

$$b_{zz} = \sum_{m=-\infty}^{-2} b_m z^{-2+m}, \tag{21}$$

$$c^z = \sum_{m=-\infty}^{1} c_m z^{1+m}. \tag{22}$$

The state on the boundary can therefore be written as a wave function

$$\psi(\cdots, b_4, b_3, b_2, c_{-1}, c_0, c_1, \cdots) \equiv \langle\psi|\cdots, b_4, b_3, b_2, c_{-1}, c_0, c_1, \cdots\rangle$$

on the configuration space parameterized by the above expansion coefficients. This wave function is given by the path integral as

$$\begin{aligned}\psi &= \mathcal{Z}\, e^{-\int_{\bar{D}}(b_{zz}\partial_{\bar{z}}c^z + c^z\partial_{\bar{z}}b_{zz}) - \int_{\delta D} dz\, b_{zz}c^z + (z\leftrightarrow\bar{z})}\\ &= \mathcal{Z}\, e^0\\ &= \mathcal{Z}.\end{aligned}$$

The boundary term in the action again ensures the equations of motion and cancels the corresponding boundary term on the disc above. Ignoring the antiholomorphic contribution for now, we may write this state in terms of the dual to the conventionally defined ghost ground state, given by

$$\langle{\downarrow}|\cdots, b_2, b_1, c_0, c_1, \cdots\rangle = 1,$$

by calculating

$$
\begin{aligned}
\langle\psi|\cdots,b_2,b_1,c_0,c_1,\cdots\rangle &= \int dc_1\ \langle\psi|\cdots,b_4,b_3,b_2,c_{-1},c_0,c_1,\cdots\rangle\ \langle c_{-1}|b_1\rangle \\
&= \int dc_{-1}\, e^{ic_{-1}b_1}\, \mathcal{Z} \\
&= \mathcal{Z}\, b_1 \\
&= \mathcal{Z}\ \langle\downarrow|\, b_{-1}\,|\cdots,b_2,b_1,c_0,c_1,\cdots\rangle\,.
\end{aligned}
$$

We find

$$\langle\psi| = \mathcal{Z}\ \langle\downarrow|\, b_1,$$

or, including the antiholomorphic contribution,

$$\langle\psi| = \mathcal{Z}\ \langle\downarrow|\, b_1\bar{b}_1.$$

We will see later that the fixed vertex operators in string theory are accompanied by a factor of

$$c^z c^{\bar{z}}.$$

We illustrate the calculation of the path integral over the disc with such an insertion at the origin

$$\psi = \int_D [dc]\,[db]\, c^z(0)\, c^{\bar{z}}(0)\, e^{-S}.$$

Again, we expand around the stationary solution $c \to c + \tilde{c}$ to get

$$
\begin{aligned}
\psi =& c_1\bar{c}_1\,\mathcal{Z} + c_1\,\langle\tilde{c}^z(0)\rangle + \langle\tilde{c}^z(0)\,\tilde{c}^{\bar{z}}(0)\rangle \\
&= c_1\bar{c}_1\,\mathcal{Z},
\end{aligned}
$$

since the one-point functions around a stationary background are zero, and the two-point function is trivially zero on the plane, since there is no $c^z c^{\bar{z}}$ term in the action.

Again temporarily ignoring the antiholomorphic contribution, we calculate

$$
\begin{aligned}
\langle\cdots,b_{-2},b_{-1},c_0,c_{-1},\cdots|\,\psi\rangle &= \int dc_1\ \langle b_{-1}|c_1\rangle\ \langle\cdots,b_{-4},b_{-3},b_{-2},c_1,c_0,c_{-1},\cdots|\,\psi\rangle \\
&= \int dc_1\, e^{ib_{-1}c_1}\, c_1\,\mathcal{Z} \\
&= \mathcal{Z} \\
&= \mathcal{Z}\ \langle\cdots,b_{-2},b_{-1},c_0,c_{-1},\cdots|\,\downarrow\rangle\,.
\end{aligned}
$$

The result is therefore

$$|\psi\rangle = \mathcal{Z}\,|{\downarrow}\rangle\,.$$

Similarly to the above calculation without insertion, the state obtained by integrating over the complement of the disc with an insertion at ∞

$$\psi = \int_{\bar{D}} [dc]\,[db]\,c^z(\infty)\,c^{\bar{z}}(\infty)\,e^{-S},$$

is proportional to

$$\langle{\downarrow}|\,.$$

The path integral over the full sphere with the two insertions is then given by integrating these states over the configuration space on the boundary

$$\begin{aligned}
\langle{\downarrow}|{\downarrow}\rangle &= \int (\cdots db_{-2}\,db_{-1}\,dc_0\,dc_{-1}\cdots)\,\langle{\downarrow}|\cdots,b_{-2},b_{-1},c_0,c_{-1},\cdots\rangle \times \\
&\qquad\qquad\qquad\qquad \times \langle\cdots,b_{-2},b_{-1},c_0,c_{-1},\cdots|\,{\downarrow}\rangle \\
&= \int (\cdots db_{-2}\,db_{-1}\,dc_0\,dc_{-1}\cdots)\,\langle{\downarrow}|\cdots,b_2,b_1,c_0,c_1,\cdots\rangle\,\langle c_1|b_{-1}\rangle\,\langle c_2|b_{-2}\rangle\cdots \\
&\qquad\qquad\qquad\qquad \times \langle\cdots,b_{-2},b_{-1},c_0,c_{-1},\cdots|\,{\downarrow}\rangle \\
&= \int (\cdots db_{-2}\,db_{-1}\,dc_0\,dc_{-1}\cdots)\,(1)\,e^{ic_1b_1}e^{ic_2b_2}\cdots(1) \\
&= 0,
\end{aligned}$$

due to the integral over c_0. We see that the path integral on the sphere with two fixed vertex insertions vanishes. It is clear that an extra insertion at a third position, proportional to, for example,

$$c^z(1)\,\bar{c}^z(1),$$

will be sufficient to provide an extra factor $c_0\bar{c}_0$ to make the path integral non-vanishing. In other words, the first non-vanishing string amplitude will be the three-point function.

7 Background-independent measures

In [1], the background dependence of the path integral measure for a scalar field ϕ was studied. The Fujikawa approach [2, 7] that we followed there depended on the existence of an a priori background metric g^{ij} for the construction of the measure, denoted $[d\phi]_g$. Varying g caused a variation in $[d\phi]_g$ that could be related to the conformal anomaly.

We then introduced a set of Pauli-Villars auxiliary fields χ with statistics opposite to that of ϕ. Due to the properties of Grassmann fields, the corresponding measure $[d\chi]_g$ transformed with opposite sign to $[d\phi]_g$ under a variation of g. As a result, the combined measure

$$[d\phi]_g[d\chi]_g = [d\phi]_{g'}[d\chi]_{g'} \equiv [d\phi]\,[d\chi]$$

was background-independent.

Something similar can be done for arbitrary tensor fields by including Pauli-Villars partners in this way. This inclusion of Pauli-Villars fields to simplify the transformation properties of the measure differs from the approach to string quantization followed by Fujikawa in [3], although individual factors contributing to the measure are defined in essentially the same way.

We start with the example of a vector field. The Fujikawa measure for a vector field can be constructed as follows. We make the space of vector fields on the plane into an inner product space by defining

$$\langle v|w\rangle_g \equiv \int \sqrt{g}\, g_{kl}\, v^k w^l.$$

We then consider an orthonormal basis $\phi_n^k(x)\,\partial_k$ of vector fields on the plane, where n is an abstract index ranging, in the two-dimensional case, over $\mathbf{N} \times \{1, 2\}$. In other words,

$$\int d^2x\, \sqrt{g}\, g_{kl}\, \phi_n^k \phi_m^l = \delta_{mn}. \tag{23}$$

Parameterizing v^k in terms of a countable set of coordinates a_n via the basis expansion

$$v^k(x) = \sum_n a_n\, \phi_n^k(x),$$

we define the measure via the semi-infinite differential form

$$[dv]_g \equiv \bigwedge_n da_n$$

on the infinite-dimensional space of fields. Since different orthonormal bases are related by unitary transformations that have Jacobian 1, the differential form is in fact independent of the choice of basis.

Since the basis ϕ_n^k depends by construction on g^{ij}, the measure is background-dependent. We may calculate the precise dependence as follows. First, note that we can project

$$a_n = \int \sqrt{g}\, g_{kl}\, \phi_n^l v^k,$$

so that

$$\delta_g a_n = \int \sqrt{g} \left(\frac{1}{2}\, g^{ab} \delta g_{ab}\, g_{kl} + \delta g_{kl} \right) \phi_n^l v^k + \int \sqrt{g}\, g_{kl}\, (\delta_g \phi_n^l)\, v^k.$$

The variation $\delta_g \phi_n^l$ of the orthonormal basis is not unique, but may be determined up to a unitary transformation, which does not affect the measure, by differentiating (23) to find

$$0 = \int \sqrt{g} \left(\frac{1}{2}\, g^{ab}\, \delta g_{ab}\, g_{kl} + \delta g_{kl} \right) \phi_n^k \phi_m^l + \int \sqrt{g}\, g_{kl} \left((\delta \phi_n^k)\, \phi_m^l + \phi_n^k\, (\delta \phi_m^l) \right)$$

A suitable choice that satisfies this is

$$\delta \phi_n^k = -\frac{1}{4}\, g^{ab}\, \delta g_{ab}\, \phi_n^k - \frac{1}{2}\, g^{kp}\, \delta g_{pq}\, \phi_n^q.$$

Inserting in the above, we find

$$\begin{aligned} \delta_g a_n &= \int \sqrt{g} \left(\frac{1}{4}\, g^{ab}\, \delta g_{ab}\, g_{kl} + \frac{1}{2}\, \delta g_{kl} \right) \phi_n^l v^k \\ &= \sum_n a_m \int \sqrt{g} \left(\frac{1}{4}\, g^{ab}\, \delta g_{ab}\, g_{kl} + \frac{1}{2}\, \delta g_{kl} \right) \phi_n^l \phi_m^k \\ &\equiv \sum_n C_{nm}\, a_m, \end{aligned}$$

so that the measure transforms as

$$\delta_g [dv]_g = \delta_g \bigwedge_n da_n = (\mathrm{Tr}\, C) \bigwedge_n da_n,$$

where

$$\mathrm{Tr}\, C = \sum_m C_{mm} = \int d^2x \sqrt{g} \left(\frac{1}{4}\, g^{ab}\, \delta g_{ab}\, g_{kl} + \frac{1}{2}\, \delta g_{kl} \right) \sum_m \phi_m^k(x)\, \phi_m^l(x) \quad (24)$$

The sum over m does not converge and has to be regulated. This may be done, for example, by inserting a heat kernel regulator, considering instead

$$\sum_m \phi^k_m(x)\, e^{\epsilon\Delta_g}\, \phi^l_m(x),$$

for small ϵ, where $\Delta_g \equiv \nabla^a\nabla_a$ is the Laplacian on the space of vector fields. These heat kernels may be calculated as, for example, in [1], and typically lead to a divergence of order $1/\epsilon$, which may be canceled by a counterterm in the action, and a finite contribution depending on the curvature that cannot be canceled by a counterterm. This contribution can be related to the conformal anomaly.

However, we can cancel the background dependence of the measure by including Pauli-Villars auxiliary vector fields. As in [1], we introduce a set of these, consisting of real commuting fields ν^k_i, each with statistics $c_i = 1$, and complex Grassmann fields ν^k_i, $\bar{\nu}^k_i$, each with statistics $c_i = -2$, satisfying

$$\sum_i c_i = -1.$$

The Grassmann fields have to be taken complex, otherwise the mass term

$$M^2\, g_{kl}\, \bar{\nu}^k \nu^l$$

would vanish for them. Due to the transformation properties of Grassmann integrals, we find, denoting the combined measure of all the Pauli-Villars fields by $[d\nu]_g$, that

$$\delta_g[d\nu]_g = \left(\sum_i c_i\right)(\mathrm{Tr}\, C)\,[d\nu]_g = -(\mathrm{Tr}\, C)\,[d\nu]_g,$$

where the sign is *opposite* to that of the original fields. Therefore

$$\delta_g\left([dv]_g \wedge [d\nu]_g\right) = \Big((\mathrm{Tr}\, C) - (\mathrm{Tr}\, C)\Big)[dv]_g \wedge [d\nu]_g = 0$$

with *any* reasonable regularization. In other words, the combined measure is background-independent, independently of the regularization scheme.

Once we include the Pauli-Villars fields, the measure does not transform anomalously. So what happened to the conformal anomaly? The burden of encoding this is now transferred to the Pauli-Villars action. To understand this, notice that the anomalous contribution to the transformation of the original measure, given by

$$\int d^2x\sqrt{g}\left(\frac{1}{4}\,g^{ab}\,\delta g_{ab}\,g_{kl}+\frac{1}{2}\,\delta g_{kl}\right)\sum_m \phi_m^k(x)\,\phi_m^l(x),$$

is just the energy-momentum contribution we would obtain from the mass terms of the Pauli-Villars fields ν

$$\begin{aligned}\delta_g\int[dv]\wedge[d\nu]\,e^{-S}&=-\frac{1}{4\pi}\int d^2x\,\sqrt{g}\,\delta g_{kl}\left\langle T_\nu^{kl}\right\rangle\\&\to-\delta_g\int d^2x\,\sqrt{g}\,\frac{1}{2}\,M^2\,g_{kl}\left\langle\bar\nu^k(x)\,\nu^l(x)\right\rangle\end{aligned}$$

in the limit as the mass tends to infinity, since then the self-contraction becomes

$$\left\langle\bar\nu^k(x)\,\nu^l(x)\right\rangle\to-\frac{2}{M^2}\sum_m\phi_m^k(x)\,\phi_m^l(x)$$

for the Grassmann-valued $(\bar\nu_i,\nu_i)$, and

$$\left\langle\bar\nu^k(x)\,\nu^l(x)\right\rangle\to\frac{1}{M^2}\sum_m\phi_m^k(x)\,\phi_m^l(x)$$

for the commuting partners $\bar\nu_i=\nu_i$. Since $\sum_i c_i=-1$, it follows that the mass terms needed to make the Pauli-Villars fields non-dynamic will now contribute the anomaly.

A similar analysis can be done for other tensor fields. For example, consider the Grassmann-valued ghost vector field $c=c^i\partial_i$. This is a vector field and, as above, we define the measure as

$$dc_1\wedge dc_2\wedge\cdots,$$

where c_n are the expansion coefficients with respect to the above orthonormal basis ϕ_n^k.

The only difference from the above analysis appears in the mass term. A non-vanishing covariant scalar mass term for a ghost c^i would be

$$\frac{1}{2}\,m^2\left(\sqrt{g}\,dx^1\wedge dx^2\right)(c,c)=\frac{1}{2}\,m^2\,\sqrt{g}\,\epsilon_{ij}\,c^ic^j,$$

and this is the form of the mass term for the Grassmann subset of the ghost Pauli-Villars partners. The bosonic Pauli-Villars partners, which have to be complex fields for this expression to be non-vanishing, have mass terms

$$\frac{1}{2}\, M^2 \left(\sqrt{g}\, dx^1 \wedge dx^2\right) (\bar{\gamma}, \gamma),$$

Note, however, that these terms are not positive definite. As a result, their Euclidean path integral $\int [d\gamma]\, e^{-S}$ does not exist. However, a consistent non-perturbative construction of the real time path integral $\int [d\gamma]\, e^{iS}$ can be done [23], which justifies the formal manipulations carried out on this object in the Euclidean picture by treating it as if it were convergent.

The anomalous variation of the ghost measure is, by construction, opposite to that of the vector measure. When the Pauli-Villars fields are included, the total measure is again invariant, and the anomaly that would otherwise have come from the measure will again have to be encoded in the Pauli-Villars energy momentum tensor.

This means that a ghost anomaly, calculated for the Pauli-Villars action, should be opposite to a vector anomaly calculated in the first part of this section, even though the mass terms have a different form. To see that this is true, we may use conformal coordinates without loss of generality, since nothing in the above construction depends on the coordinate choice. The mass contribution for a vector partner is

$$\frac{1}{2}\, M^2 \int d^2x\, \sqrt{g}\, g_{z\bar{z}}\, \bar{\nu}^{\bar{z}} \nu^z = \frac{1}{2}\, M^2 \int d^2x\, g^2_{z\bar{z}}\, \bar{\nu}^{\bar{z}} \nu^z,$$

and for a ghost partner it is

$$\frac{1}{2}\, M^2 \int d^2x\, \sqrt{g}\, \sqrt{g}\, \epsilon_{z\bar{z}}\, \bar{\gamma}^{\bar{z}} \gamma^z = \frac{1}{2}\, M^2 g^2_{z\bar{z}} \bar{\gamma}^{\bar{z}} \gamma^z,$$

while an arbitrary variation of the metric (not necessarily conformal) gives for a vector partner

$$\begin{aligned} \frac{1}{2}\, M^2 \delta \left(\sqrt{g}\, g_{z\bar{z}}\right) \langle \bar{\nu}^{\bar{z}} \nu^z \rangle &= \frac{1}{2}\, M^2 \left(\frac{1}{2} \sqrt{g} \left(g^{kl}\, \delta g_{kl}\right) g_{z\bar{z}} + \sqrt{g}\, \delta g_{z\bar{z}} \right) \langle \bar{\nu}^{\bar{z}} \nu^z \rangle \\ &= M^2\, g_{z\bar{z}}\, \delta g_{z\bar{z}}\, \langle \bar{\nu}^{\bar{z}} \nu^z \rangle \end{aligned}$$

and for a ghost partner

$$\begin{aligned} \frac{1}{2}\, M^2\, \delta g\, \epsilon_{z\bar{z}}\, \langle \bar{\gamma}^{\bar{z}} \gamma^z \rangle &= \frac{1}{2}\, M^2\, \delta \left(g^2_{z\bar{z}}\right) \epsilon_{z\bar{z}}\, \langle \bar{\gamma}^{\bar{z}} \gamma^z \rangle \\ &= M^2\, g_{z\bar{z}}\, \delta g_{z\bar{z}}\, \langle \bar{\gamma}^{\bar{z}} \gamma^z \rangle \end{aligned}$$

Since, in the limit of large M, the mass terms become dominant in the action, and since these coincide for the two cases above, we find the same contribution except for a sign. In other words, the ghost anomaly, as encoded in the unusual ghost Pauli-Villars mass term, is indeed opposite to the vector anomaly.

The measure on the space of metrics g^{ij} is particularly important. We construct the background-independent path integral measure over g^{ij} as above as follows. Choose a background metric $\bar{g}^{ij}$, and let $[dg]_{\bar{g}}$ be the Fujikawa measure on the space of metrics with respect to $\bar{g}^{ij}$. In other words, we expand, similar to the above, in a basis on the space of symmetric tensors g^{ij} orthonormalized in the obvious inner product with respect to $\bar{g}^{ij}$, and define the measure, as above, as a differential form in terms of the expansion coefficients. This measure depends on $\bar{g}^{ij}$, and is therefore not yet suitable for a background-independent theory. To obtain a background-independent measure, introduce a set of Pauli-Villars auxiliary fields, denoted $\{\gamma^{ij}, \bar{\gamma}^{ij}\}$ that are either real commuting with $\bar{\gamma} = \gamma$ and with statistics 1, or complex Grassmann with statistics -2, whose combined statistics is equal to -1 to cancel that of the original g^{ij} (the Grassmann γ are symmetric, complex-valued matrices, not hermitian matrices). As above, the combined measure is then independent of $\bar{g}$, independently of which regularization we choose. In particular,

$$[dg]_{\bar{g}} \wedge [d\gamma]_{\bar{g}} = [dg]_g \wedge [d\gamma]_g,$$

so that the combined measure is background independent.

A slight complication that has to be kept in mind is that the range of an integration over the metrics is not unconstrained, as was the case for scalar or vector fields, since we have to ensure that the metric does not become degenerate or change signature. In the two-dimensional case, we will manage to avoid the need for integrating over the metrics, so this issue will not affect the analysis that follows, but it should be kept in mind in higher dimensions.

8 Physical state conditions

In the operator formalism, the physical state condition on the vertex operators $\hat{V}$ are related to their anomalous dimension. In the current formalism, the Ward identity (9) for the insertion $V = \exp\left(ik\left(X + \sum \eta_i \chi_i\right)\right)$ is that of a scalar, and the anomaly is encoded in the non-vanishing contact term matrix elements of $T_{w\bar{w}} V_z$. Given this difference, it is very instructive to derive the physical state condition on V in the current coordinate-independent formalism.

Throughout this section we will use the covariant form of the path integral derived later in (34).

Consider a path integral with n vertex insertions, where the path integral includes an integral over the positions x_i, $i = 1, \ldots, n$ of the insertions. By a gauge transformation consisting of diffeomorphisms and Weyl transformations, we can bring the metric to a chosen form $\hat{g}$, modulo possible global obstructions that do not occur on the plane or the sphere. By further conformal transformations (which are compositions of diffeomorphisms and conformal transformations leaving the metric invariant), we may fix the positions of $m \leq n$ of the x_i to be at chosen points $\hat{x}_i$, $i = 1, \ldots, m$, where the value of m depends on the topology. On the plane, m is one, since v may go to a constant at infinity, whereas on the sphere, a Möbius transformation may be used to fix $m = 3$ positions in this manner.

Consider then a family of gauge fixing functions

$$\delta(g^{ij} - \hat{g}^{ij})\, \delta(x_1 - \hat{x}_1) \cdots \delta(x_m - \hat{x}_m),$$

indexed by $(\hat{g}^{ij}, \hat{x}_1, \ldots, \hat{x}_m)$. Here g^{ij} and x_i are the metric and positions over which we integrate in the path integral, and $\hat{g}^{ij}$ and $\hat{x}_i$ are a fixed metric and fixed positions. Gauge fixing will discussed in great detail in section 10 and in the BRST context, but for the purposes of this section we will only need the familiar covariant form of the full action and the resulting form (33) of the fixed vertices.

Any physical quantity has to be independent of the hatted quantities determining the gauge fixing function. We will show how this requirement leads to the physical state conditions on the vertex insertions, and later relate it to a covariant version of BRST invariance in the quantum field theory.

Consider a variation $(\delta\hat{g}^{ij}, \delta\hat{x}_1 \ldots)$ of the fixed metric and positions under a gauge transformation consisting of a combined reparametrization $v^i\partial_i$ and Weyl transformation.

The next step is to write the gauge fixing function $\delta(g^{ij} - \hat{g}^{ij})$ in a more convenient form. To do this, we introduce a Langrange multiplier field B_{ij}, together with its set of Pauli-Villars partners, conveniently denoted by $\{\beta_{ij}, \bar{\beta}_{ij}\}$. As above, their combined measure is background-independent

$$[dB]_{\bar{g}} \wedge [d\beta]_{\bar{g}} = [dB]_g \wedge [d\beta]_g$$

and we may write

$$
\begin{aligned}
\int [dg]_{\bar g}\, \delta(g^{ij} - \hat g^{ij}) &= \int [dg]_{\bar g} \wedge [dB]_{\bar g} \wedge [d\beta]_{\bar g} \wedge [d\gamma]_{\bar g} \\
&\qquad \times\ \exp i \left\{ \langle B,\, (g - \hat g)\rangle_{\bar g} + \frac{1}{2}\left(\langle \beta,\, \gamma\rangle_{\bar g} + \left\langle \bar\gamma,\, \bar\beta \right\rangle_{\bar g} \right) \right\} \\
&= \int [dg]_{\bar g} \wedge [dB]_{\bar g} \wedge [d\beta]_{\bar g} \wedge [d\gamma]_{\bar g} \\
&\qquad \times \exp i \int d^2x \sqrt{\bar g}\, \left\{ B_{ij}\,(g^{ij} - \hat g^{ij}) + \frac{1}{2}\left(\bar\beta_{ij}\gamma^{ij} + \bar\gamma^{ij}\,\beta_{ij} \right) \right\},
\end{aligned}
$$

since the integral over β and γ gives 1. This formula is an easy consequence of the definition of the Fujikawa measures in terms of orthonormalized modes with respect to $\bar g^{ij}$. Note that the background metric $\bar g$ is different from the metric $\hat g$ appearing in the gauge fixing function. Since the result of integrating over β and γ is 1 independently of $\bar g$, it follows that the remaining expression

$$
\int [dg]_{\bar g}\, [dB]_{\bar g}\, \exp i \int d^2x \sqrt{\bar g}\, B_{ij}\,(g^{ij} - \hat g^{ij})
$$

is itself independent of $\bar g$. It is indeed not difficult to show this explicitly by demonstrating that the variation of the measure $[dg]_{\bar g}[dB]_{\bar g}$ under a change of $\bar g$ cancels the variation of the integrand. Indeed, one finds

$$
\begin{aligned}
&\delta_{\bar g} \int [dg]_{\bar g}[dB]_{\bar g}\, \exp i \int d^2x \sqrt{\bar g}\, B_{ij}\,(g^{ij} - \hat g^{ij}) \\
&\quad = (\operatorname{Tr} C + \operatorname{Tr} C - 2 \operatorname{Tr} C) \int [dg]_{\bar g}[dB]_{\bar g}\, \exp i \int d^2x \sqrt{\bar g}\, B_{ij}\,(g^{ij} - \hat g^{ij}) \\
&\quad = 0.
\end{aligned}
$$

The first two terms come from the variations of the two measures, as in the previous section, while the third comes from evaluating the expectation value obtained when varying the integrand. Here

$$
C_{mn} \equiv \frac{1}{2} \int \delta\left(\sqrt{\bar g}\, \bar g_{ik}\, \bar g_{jl} \right) \phi^{kl}_m \phi^{ij}_n,
$$

for a $\bar g$-orthonormal basis ϕ^{kl}_m of symmetric tensors.

However, it should be noted that this argument only works for the vacuum amplitude. Insertions containing g^{ij} in the amplitude will in general cause the invariance to break down due to additional contractions with B^{ij}. We therefore have to keep the $\bar{g}$ dependence in the action separate. In particular, we *cannot* use this to set $\bar{g}^{ij} = g^{ij}$ for general amplitudes.

The full background-independent measure is given by

$$\begin{aligned} d\mu_{\bar{g}} &\equiv [dg]_{\bar{g}} \wedge [d\gamma]_{\bar{g}} \wedge [dB]_{\bar{g}} \wedge [d\beta]_{\bar{g}} \wedge [dX]_{\bar{g}} \wedge [d\chi]_{\bar{g}} \wedge \text{ghost} \\ &= d\mu_g \\ &\equiv d\mu \end{aligned}$$

where the ghost contribution includes the Pauli-Villars partners for each ghost, to be considered more carefully in section 10.

The path integral will be independent of the gauge fixing function if

$$\begin{aligned} 0 &= (\delta_{\hat{g}} + \delta_{\hat{x}_i}) \int d\mu \, \left(\sqrt{g}\, \epsilon_{kl}\, c^k c^l \, V\right)(\hat{x}_1) \cdots e^{-S + i\int d^2x \left\{B_{ij}\,(g^{ij} - \hat{g}^{ij}) + \frac{1}{2}\left(\bar{\beta}_{ij}\gamma^{ij} + \bar{\gamma}^{ij}\,\beta_{ij}\right)\right\}} \\ &= -i \int d\mu \, \tilde{V}(\hat{x}_1) \cdots e^{-\tilde{S}} \int d^2x \, \sqrt{\bar{g}}\, B_{ij}\, \delta\hat{g}^{ij} \\ &\quad - v^i \hat{\partial}_i \left\langle \tilde{V}(\hat{x}_1) \cdots \right\rangle_{\tilde{S}}, \end{aligned}$$

where

$$\begin{aligned} \tilde{S} &\equiv S - i \int d^2x \, \sqrt{\bar{g}} \left\{ B_{ij}\,(g^{ij} - \hat{g}^{ij}) + \frac{1}{2}\left(\bar{\beta}_{ij}\gamma^{ij} + \bar{\gamma}^{ij}\,\beta_{ij}\right) \right\}, \\ \tilde{V} &\equiv \sqrt{g}\, \epsilon_{kl}\, c^k c^l \, V \end{aligned}$$

and the ghost factors accompanying the fixed vertices come from the Faddeev-Popov determinant and are derived in section 10. The action $\tilde{S}$ is in fact derived in detail later and is given by equation (34). If V is a scalar, then $\tilde{V}$ will be a coordinate-independent scalar quantity, since the prefactor is simply the application

$$\left(\sqrt{g}\, dx^1 \wedge dx^2\right)(c, c)$$

of a coordinate-independent bilinear density to the pair (c, c).

Remembering the definition of the energy momentum-tensor, we have

$$\frac{\delta}{\delta g^{ij}}\, e^{-\tilde{S}} = i\sqrt{\bar{g}}\, B_{ij}\, e^{-\tilde{S}} + \frac{1}{4\pi}\, \sqrt{g}\, T_{ij}\, e^{-\tilde{S}},$$

where T_{ij} denotes the part of the energy-momentum tensor of $\tilde{S}$ independent of B. Notice that the terms in β and γ are independent of g^{ij} and therefore do not contribute to the energy-momentum tensor.

We obtain the condition

$$
\begin{aligned}
0 &= \int d\mu\, \tilde{V}(\hat{x}) \cdots \int d^2x \left(-\frac{\delta}{\delta g^{ij}(x)} + \frac{1}{4\pi} \sqrt{g}\, T_{ij} \right) e^{-\tilde{S}}\, \delta \hat{g}^{ij} - v^i \hat{\partial}_i \left\langle \tilde{V} \cdots \right\rangle_{\tilde{S}} \\
&= \int d^2x \int d\mu\, e^{-\tilde{S}} \left(\frac{\delta}{\delta g^{ij}(x)} \tilde{V}(\hat{x}) + \frac{1}{4\pi} \sqrt{g}\, T_{ij}(x)\, \tilde{V}(\hat{x}) \right) \delta \hat{g}^{ij}(x) \cdots - v^i \hat{\partial}_i \left\langle \tilde{V} \cdots \right\rangle_{\tilde{S}} \\
&= \left\langle \frac{\delta \tilde{V}}{\delta g^{ij}} \cdots \right\rangle_{\tilde{S}} \delta \hat{g}^{ij}(\hat{x}) + \frac{1}{4\pi} \int d^2x \left\langle \sqrt{g}\, T_{ij}(x)\, \tilde{V}(\hat{x}) \cdots \right\rangle_{\tilde{S}} \delta \hat{g}^{ij}(x) - v^i \hat{\partial}_i \left\langle \tilde{V} \cdots \right\rangle_{\tilde{S}} \\
&= \left\langle \frac{\delta \tilde{V}^{\hat{g}}}{\delta \hat{g}^{ij}} \cdots \right\rangle_{S(\hat{g})} \delta \hat{g}^{ij}(\hat{x}) + \frac{1}{4\pi} \int d^2x\, \sqrt{\hat{g}} \left\langle T^{\hat{g}}_{ij}(x)\, \tilde{V}^{\hat{g}}(\hat{x}) \cdots \right\rangle_{S(\hat{g})} \delta \hat{g}^{ij}(x) - v^i \hat{\partial}_i \left\langle \tilde{V}^{\hat{g}} \cdots \right\rangle_{S(\hat{g})}
\end{aligned}
$$

where we have used translation invariance of the path integral measure with respect to g^{ij} to perform a partial integration in the second step. In the last step, we have performed the path integral over $[dg] \wedge [d\gamma] \wedge [dB] \wedge [d\beta]$, which fixes $g = \hat{g}$, and we are left with the path integral

$$
\langle \cdots \rangle_{S(\hat{g})} \equiv \int [dX]_{\hat{g}} \wedge [d\chi]_{\hat{g}} \wedge (\text{ghost})\, e^{-S(\hat{g})} \, (\cdots),
$$

where the measure is in fact independent of $\hat{g}$. We will henceforth drop the $S(\hat{g})$ subscripts on expectation values.

But if we decompose the variation of $\hat{g}^{ij}$ in the gauge directions as

$$
\delta \hat{g}^{ij} = -\nabla^i v^j - \nabla^j v^i - 2\, \delta\omega\, g^{ij},
$$

then the v-dependent part of the above condition is exactly zero by the Ward identity (7, 8), and all that remains is the $\delta\omega$-dependent part. The physical state condition becomes simply

$$
0 = \left\langle \frac{\delta \tilde{V}}{\delta \hat{g}^{ij}}\, \hat{g}^{ij} \cdots \right\rangle \delta\omega(\hat{x}) + \frac{1}{4\pi} \int d^2x\, \sqrt{\hat{g}} \left\langle T_i{}^i(x)\, \tilde{V}(\hat{x}) \cdots \right\rangle \delta\omega(x) \tag{25}
$$

This condition has an intuitively reasonable interpretation. It requires that the path integral with insertion be invariant under a local Weyl rescaling, generated by $T_i{}^i$, of the metric $\hat{g}^{ij}$ that indexes the choice of gauge-fixing function. It is also interesting to see that invariance with respect to diffeomorphisms is automatic, and puts no further constraints on $\tilde{V}$.

The simplest scalar insertion satisfying this gauge invariance condition, for certain values of k, is the tachyon

$$\tilde{V} = \sqrt{g}\,\epsilon_{kl}\,c^k c^l\,e^{ik\tilde{X}} \to c^z c^{\bar{z}} e^{ik\tilde{X}},$$

where $\tilde{X}$ stands for the combination

$$\tilde{X} \equiv X + \sum_i \eta_i \chi_i.$$

No regularization is needed for the ghost prefactor, since the self-contraction of $c^k c^l$ is trivially zero due to the absence of any $c^k c^l$ term in the massless action for the ghosts.

For simplicity, we will concentrate on variations of the flat metric on the plane $\hat{g}^{z\bar{z}} = 2$, $\hat{g}^{zz} = 0 = \hat{g}^{\bar{z}\bar{z}}$. Inserting the vertex into the condition (25), and using

$$\frac{\partial\sqrt{g}}{\partial g^{ij}} = -\frac{1}{2}\sqrt{g}\,g_{ij},$$

we obtain

$$-\delta\omega(z)\left\langle \tilde{V}(z)\cdots\right\rangle + \frac{1}{4\pi}\int d^2w\,\left\langle (T_{w\bar{w}} + T_{\bar{w}w})\,c^z c^{\bar{z}} e^{ik\tilde{X}(z)}\cdots\right\rangle\,(2\,\delta\omega(w)) = 0.$$

We have already calculated the contraction of $T_{w\bar{w}}$ with $e^{ik\tilde{X}}$ in section 5. We still have to consider the contribution

$$\langle T_{w\bar{w}}\,c^z c^{\bar{z}}\rangle\;e^{ik\tilde{X}}.$$

Since the $\{b, c\}$ theory is massless, so that $T_{w\bar{w}}(b, c) = 0$, and no Pauli-Villars contractions are involved here, this expression trivially vanishes.

Using the explicit result from section 5,

$$T_{w\bar{w}}\,e^{ik\tilde{X}_z} = \pi\,\frac{\alpha' k^2}{4}\,\delta^2(w - z)\,e^{ik\tilde{X}_z} + \cdots\,, \tag{26}$$

we find

$$\left(\frac{1}{2} - \frac{1}{2\pi}\,\pi\,\frac{\alpha' k^2}{4}\right)\left\langle \tilde{V}\cdots\right\rangle = 0$$

so that we obtain the usual tachyon mass-shell condition

$$k^2 = \frac{4}{\alpha'}.$$

Next we consider the graviton vertex, given by the worldsheet scalar

$$\begin{aligned}\tilde{V} &= \frac{1}{4}\sqrt{g}\,\epsilon_{kl}c^k c^l\, g^{ab}\, e^{ij}\,\partial_a\tilde{X}_i\partial_b\tilde{X}_j\, e^{ik\tilde{X}} \\ &\rightarrow c^z c^{\bar{z}}\, e^{ij}\,\partial_z\tilde{X}_i\,\partial_{\bar{z}}\tilde{X}_j\, e^{ik\tilde{X}}\end{aligned}$$

where e^{ij} is a spacetime polarization. To evaluate the first terms in the condition (25), we calculate

$$g^{ij}\,\frac{\partial\sqrt{g}\, g^{ab}}{gij} = \sqrt{g}\left(-\frac{1}{2}\, g_{ij}\, g^{ab} + \delta^a_i\delta^b_j\right) g^{ij} = 0,$$

so that the gauge invariance condition becomes simply

$$\frac{1}{4\pi}\int d^2w\,\left\langle (T_{w\bar{w}} + T_{\bar{w}w})\,\tilde{V}(z)\cdots\right\rangle (2\,\delta\omega(w)) = 0. \qquad (27)$$

To verify the consistency of our covariant approach, we will now calculate the contact terms in this condition from first principles. The calculation is complicated, and this is not the recommended way of doing things. In the next section, we will discuss the relationship between the above physical state condition and the familiar Virasoro conditions. The latter may be computed using familiar methods in the literature, and the results may then be used to infer the contact terms indirectly.

With this in mind, let us return to the direct calculation. In holomorphic coordinates, where

$$\tilde{V}(z) \equiv c^z c^{\bar{z}}\,\partial_z\left(X(z) + \sum_i \eta_i\chi_i(z)\right)\partial_{\bar{z}}\left(X(z) + \sum_i \eta_i\chi_i(z)\right) e^{ik(X(z)+\sum\eta_i\chi_i(z))}$$

the graviton condition becomes

$$
\begin{aligned}
&T_{w\bar{w}}\,\tilde{V}(z)\\
&\quad = \frac{\pi}{2}\left(m^2X^2 + \sum_j M_j^2\chi_j^2\right) V(z)\\
&\quad = \frac{\pi}{2}\,\frac{(ik)^2}{2!}\Big(m^2\left\langle X^2(w)\,X^2(z)\right\rangle + \sum_i \eta_i^2 M_i^2\left\langle \chi_i^2(w)\,\chi_i^2(z)\right\rangle\Big) V(z)\\
&\qquad + \frac{\pi}{2}\,ik\Big(m^2\ \left\langle X^2(w)\,(\partial_z X)\,X(z)\right\rangle + \sum_i \eta_i^2 M_i^2\ \left\langle \chi_i^2(w)\ (\partial_z \chi_i)\,\chi_i(z)\right\rangle\Big)\\
&\qquad\quad \times \partial_{\bar{z}}\left(X(z) + \sum_i \eta_i\chi_i(z)\right) e^{ik\left(X(z)+\sum_i \eta_i\chi_i\right)}\\
&\qquad + (z \leftrightarrow \bar{z})\\
&\qquad + \cdots .
\end{aligned} \tag{28}
$$

In the above, contributions from single contractions are absent, since these are either proportional to

$$\frac{m^2}{z-w}\,\partial_{\bar{z}}X \to 0,$$

as $m \to 0$ for the matter fields, or proportional to

$$M_i^2\,\partial_z K_0(M_i|w-z|)\,\partial_{\bar{z}}\chi_i \to 2\pi\,\partial_z\delta^2(w-z)\,\partial_{\bar{z}}\chi_i,$$

as $M_i \to \infty$, but matrix elements of $\chi_i(z)$ vanish in this limit as long as no other insertions approach the point z. We have also dropped double contractions of the form $m^2\,\langle X^2(w)\,\partial_z X\partial_{\bar{z}}X\rangle$ proportional to

$$m^2\,\frac{1}{w-z}\ \frac{1}{\bar{w}-\bar{z}} \to 0,$$

and the corresponding Pauli-Villars contributions, proportional to

$$M_i^2\,\partial_z K_0(M_i|w-z|)\,\partial_{\bar{z}}K_0(M_i|w-z|) \to (2\pi)^2\,\partial_z\delta^2(w-z)\ \partial_{\bar{z}}\delta^2(0) = 0$$

as $M_i \to \infty$.

The first contraction in (28) was calculated in section 5 and is equal to

$$\frac{\pi\alpha' k^2}{4}\,\delta^2(w-z).$$

We calculate the second contraction in (28) as follows, where we abbreviate $\bar{k} \equiv k_1 - ik_2$,

$$
\begin{aligned}
&\frac{\pi}{2}\, m^2 \, \left\langle X^2(w)\, (\partial_z X)\, X(z) \right\rangle + PV \\
&\quad = \frac{\pi}{2} \cdot m^2 \cdot 2 \cdot \frac{i}{2} \int \frac{d^2p}{(2\pi)^2}\, e^{-ipx} \int \frac{d^2k}{(2\pi)^2}\, \frac{\bar{k}}{(k^2+m^2)\,((p-k)^2+m^2)} + PV \\
&\quad = \frac{\pi}{2}\, m^2\, i\, \frac{1}{4\pi} \int \frac{d^2p}{(2\pi)^2}\, e^{-ipx} \int_0^1 dx\, \frac{(1-x)\bar{p}}{x(1-x)\, p^2 + m^2} + PV \\
&\quad = \frac{\pi}{2}\, m^2\, i\, \frac{1}{4\pi} \int \frac{d^2p}{(2\pi)^2}\, e^{-ipx} \int_0^{1/2} dx\, \frac{\bar{p}}{x(1-x)\, p^2 + m^2} + PV \\
&\quad = \frac{\pi}{2}\, \frac{i}{4\pi}\, 2 \int \frac{d^2p}{(2\pi)^2}\, e^{-ipx} \int_{2\mu}^{\infty} \frac{d\mu}{\mu^2}\, \frac{m^2}{\sqrt{1-4m^2/\mu^2}}\, \frac{\mu^2\bar{p}}{p^2+\mu^2} + PV \\
&\quad = \frac{\pi}{2}\, \frac{i}{4\pi}\, \frac{2}{8} \int \frac{d^2p}{(2\pi)^2}\, e^{-ipx} \int_1^{\infty} \frac{d\nu}{\nu^2}\, \frac{1}{\sqrt{\nu^2-1}}\, \frac{4\, m^2\nu^2\bar{p}}{p^2+4m^2\nu^2} + PV \\
&\quad \to \left(\sum_i \eta_i^2\right) \frac{\pi}{2}\, \frac{i}{4\pi}\, \frac{2}{8} \int \frac{d^2p}{(2\pi)^2}\, e^{-ipx} \int_1^{\infty} \frac{d\nu}{\nu^2}\, \frac{1}{\sqrt{\nu^2-1}}\, \bar{p} \\
&\quad = \left(\sum_i \eta_i^2\right) \frac{\pi}{2}\, \frac{i}{4\pi}\, \frac{2}{8} \int \frac{d^2p}{(2\pi)^2}\, e^{-ipx}\, \frac{1}{2}\, B\left(\tfrac{1}{2}, 1\right) \bar{p} \\
&\quad = (-1)\, \frac{\pi}{2}\, \frac{i}{4\pi}\, \frac{2}{8} \int \frac{d^2p}{(2\pi)^2}\, e^{-ipx}\, \bar{p} \\
&\quad = -\frac{\pi}{2}\, \frac{i}{4\pi}\, \frac{2}{8}\, 2i\, \partial_z \delta^2(w-z) \\
&\quad = \frac{1}{16}\, \partial_z \delta^2(w-z).
\end{aligned}
$$

The arrow indicates the limit $m \to 0$ and $M_i \to \infty$, in which the matter contribution vanishes an the Pauli-Villars contribution simplifies. As usual, changes of variables performed in the above integrals are justified by their convergence when the Pauli-Villars terms are included.

Inserting this in (28), we finally get

$$
\begin{aligned}
\left\langle T_{w\bar{w}} \tilde{V}_z \cdots \right\rangle = \frac{\pi\alpha' k^2}{4}\, \delta^2(w-z) \left\langle \tilde{V}_z \cdots \right\rangle &+ \frac{\pi\alpha' ik}{8}\, \partial_z \delta^2(w-z) \left\langle c^z c^{\bar{z}}\, \partial_{\bar{z}} \tilde{X}\, e^{ik\tilde{X}} \cdots \right\rangle \\
&+ \frac{\pi\alpha' ik}{8}\, \partial_{\bar{z}} \delta^2(w-z) \left\langle c^z c^{\bar{z}}\, \partial_z \tilde{X}\, e^{ik\tilde{X}} \cdots \right\rangle .
\end{aligned}
$$

Inserting in the gauge invariance condition (27), we find

$$0 = \frac{\pi\alpha' k^2}{4}\,\delta\omega\left\langle \tilde{V}\cdots\right\rangle - \frac{\pi\alpha'\, ie^{ij}k_j}{8}\,(\partial_z\delta\omega)\left\langle c^z c^{\bar{z}}\,\partial_{\bar{z}}\tilde{X}_i\, e^{ik\tilde{X}}\cdots\right\rangle - \frac{\pi\alpha'\, ie^{ij}k_i}{8}\,(\partial_{\bar{z}}\delta\omega)\left\langle c^z c^{\bar{z}}\,\partial_{\bar{z}}\tilde{X}_j\, e^{ik\tilde{X}}\cdots\right\rangle.$$

Since $\delta\omega$ is an arbitrary function, we find the following conditions on the momentum and the polarization.

$$0 = k^2,$$
$$0 = e^{ij}k_j = e^{ij}k_i.$$

These are the usual mass shell and polarization conditions for the graviton vertex.

9 Correspondence with Virasoro conditions

The physical state condition (25)

$$0 = \left\langle \frac{\delta\tilde{V}}{\delta\hat{g}^{ij}}\,\hat{g}^{ij}\cdots\right\rangle\delta\omega(\hat{x}) + \frac{1}{4\pi}\int d^2x\,\sqrt{\hat{g}}\left\langle T_i{}^i(x)\,\tilde{V}(\hat{x})\cdots\right\rangle\delta\omega(x) \qquad (29)$$

was derived by requiring that the functional integral be independent of the choice of gauge fixing function. This condition was physically and mathematically well motivated from first principles in the functional integral approach, but does not obviously resemble the more familiar Virasoro conditions on physical states. In particular, as we remarked, in (29) only Weyl invariance led to constraints on the physical insertions. Diffeomorphism invariance was automatic and did not contribute any constraints. In the operator formalism, the Virasoro constraints are usually motivated by a requirement of invariance under conformal transformations, which only correspond to a subclass of Weyl transformations composed with specific diffeomorphisms. It is therefore instructive to relate the two approaches.

In this section we will show that our constraint (29) in fact implies the Virasoro conditions.

To see this, consider $\delta\omega$ of the special form

$$\delta\omega = -\frac{1}{2}\,\nabla_i v^i,$$

where v^i is a vector field that is zero at $\hat{x}$, conformal in a neighbourhood D of $\hat{x}$, and sufficiently well-behaved at infinity. On the region D where v is conformal, we have, by definition of conformality,

$$\delta\omega\, g^{ij} = -\frac{1}{2}\left(\nabla^i v^j + \nabla^j v^i\right) = \frac{1}{2}\, h^{ij}.$$

For this choice of $\delta\omega$, the condition (29) becomes

$$\begin{aligned}
0 &= \left\langle \frac{\delta \tilde{V}}{\delta \hat{g}^{ij} \cdots} \right\rangle h^{ij}(\hat{x}) + \frac{1}{4\pi}\int_D d^2x\, \sqrt{\hat{g}}\left\langle T_{ij}(x)\, \tilde{V}(\hat{x}) \cdots \right\rangle h^{ij}(x) \\
&\quad + \frac{1}{4\pi}\int_{\bar{D}} d^2x\, \sqrt{\hat{g}}\left\langle T_i{}^i(x)\, \tilde{V}(\hat{x}) \cdots \right\rangle 2\,\delta\omega(x) \\
&= \left\langle \frac{\delta \tilde{V}}{\delta \hat{g}^{ij}} \right\rangle h^{ij}(\hat{x}) + \frac{1}{4\pi}\int_D d^2x\, \sqrt{\hat{g}}\left\langle T_{ij}(x)\, \tilde{V}(\hat{x}) \cdots \right\rangle h^{ij}(x),
\end{aligned}$$

by virtue of the region of integration, since $T_i{}^i(x)\, \tilde{V}(\hat{x})$ can contribute at most contact terms. Here $\bar{D}$ denotes the complement of D.

But for the restricted region of integration, this is almost the Ward identity (9), remembering that v^i is here chosen zero at $\hat{x}$. Subtracting (9) from this, the condition becomes

$$0 = \frac{1}{4\pi}\int_{\bar{D}} d^2x\, \sqrt{\hat{g}}\left\langle T_{ij}(x)\, \tilde{V}(\hat{x}) \cdots \right\rangle h^{ij}(x).$$

In holomorphic coordinates, this gives

$$0 = \int_{\bar{D}} d^2w\, \left\langle \left(\partial_{\bar{w}} v^w\, T_{ww} + \partial_w v^{\bar{w}}\, T_{\bar{w}\bar{w}}\right) \tilde{V}(z) \cdots \right\rangle .$$

Again, due to the region of integration, contact terms coming from $T_{w\bar{w}}\, \tilde{V}(z)$ do not contribute and have been dropped. A partial integration finally gives the physical state condition

$$0 = \frac{1}{2\pi i}\left\langle \oint_{\partial D} \left(dw\, v^w\, T_{ww} - d\bar{w}\, v^{\bar{w}}\, T_{\bar{w}\bar{w}}\right) c\bar{c}\, V(z) \cdots \right\rangle . \tag{30}$$

Given the basis

$$v^w \in \left\{ (w-z)^{n+1} \,|\, n \geq 0 \right\},$$

of holomorphic vector fields on D, this is seen to coincide with the usual Virasoro conditions. We remind the reader that the contraction with $c\bar{c}$ reproduces the usual a-correction to L_0.

The Virasoro conditions are preferable for actual calculations, since they avoid the need for calculating complicated contact terms. The above arguments can also be turned around to obtain certain contact terms from standard operator calculations, thus avoiding the rather complicated direct loop integral calculations of the previous section.

We have shown that the condition (29) implies the Virasoro conditions. It might seem that the converse is not necessarily true, since the $\delta\omega$ in (29) is arbitrary, whereas the class of $\delta\omega$ needed to derive the Virasoro conditions were harmonic on D, i.e., $\partial\bar{\partial}\,\delta\omega = 0$. Comparing with the graviton condition in the previous section, we can see that the difference would matter for vertices that have mixed derivatives of X, the simplest of which is $(\partial\bar{\partial}X)\,e^{ikX}$, which would then give rise to terms proportional to $\partial\bar{\partial}\,\delta\omega$ in the physical state condition. However, these vertices vanish trivially by the equation of motion, and are therefore of no interest.

10 A covariant BRST approach

The BRST action for a generally covariant theory has subtleties that are sometimes overlooked, and we will derive it carefully, obtaining an action that is different in important ways from the usual one. In particular, the need for a fixed background metric $\bar{g}^{ij}$ in carrying out the Faddeev-Popov procedure will be elucidated, and the independence of the resulting theory of this choice will be discussed.

For simplicity, we work on the plane, where the metric can be gauge-fixed completely and there are no remaining moduli. The aspects of the analysis that we would like to emphasize here do not involve the moduli.

We apply the Faddeev-Popov and BRST procedures to the case at hand [19, 20, 21, 22, 7]. Let $\bar{g}_{ij}$ be an arbitrary background metric used for defining the measure as in section 7. The first ingredient we need is a functional integration measure that is invariant under the gauge symmetries of the system. In the current case, the symmetries are diffeomorphism invariance and conformal invariance. By our discussion in section 7, both of these are preserved once we include Pauli-Villars partners for all fields. The invariant measure is therefore

$$[dg]_{\bar{g}}\,[dB]_{\bar{g}}\,[dX]_{\bar{g}}\,[d\gamma]_{\bar{g}}\,[d\beta]_{\bar{g}}\,[d\chi]_{\bar{g}}.$$

As we discussed already, this measure is independent of the background metric $\bar{g}$.

Let $\bar{\phi}_n^{kl}$ be an orthonormal basis of metric fields with respect to the inner product

$$\left\langle \bar{\phi}_n, \bar{\phi}_m \right\rangle_{\bar{g}} = \int \sqrt{\bar{g}}\,\bar{\phi}_n^{ij}\,\bar{g}_{ik}\bar{g}_{jl}\,\bar{\phi}_m^{kl}.$$

We note that $\tilde{g}^{ij}$ is different from the metric $\hat{g}_{ij}$ occurring in the gauge fixing function $\delta(g^{ij} - \hat{g}^{ij})$. Now consider the following set of constraint functions, which are just the components of the metric field in the above basis.

$$\bar{\chi}_n(g) \equiv \left\langle g, \bar{\phi}_n \right\rangle_{\bar{g}} = \int \sqrt{\bar{g}}\,g^{ij}\,\bar{g}_{ik}\bar{g}_{jl}\,\bar{\phi}_n^{kl}.$$

The Faddeev-Popov procedure instructs us to calculate the determinant of the matrix

$$\bar{h}_m(\bar{\chi}_n)$$

where $\bar{h}_m$ is a fixed basis of generators of the symmetry group. We may choose $\{\bar{h}_m\} = \{\bar{v}_p^i\} \cup \{\bar{\omega}_q\}$, where $\bar{v}_p^i$ is a $\bar{g}$-orthonormal basis of vector fields generating diffeomorphisms, and $\bar{\omega}_q$ is a $\bar{g}$-orthonormal basis of scalar fields generating Weyl rescalings.

Note that the Faddeev-Popov procedure requires the basis of generators to be fixed, independent of the variable of integration g_{ij}. We therefore could not have used g_{ij} to define them. Instead, an orthonormal basis with respect to the fixed background metric $\bar{g}_{ij}$ was used, but any fixed basis would have been acceptable. A different choice would lead at most to a different constant overall factor in the Haar measure on the group manifold. This factor would be gauge slice independent and would not affect physical expectation values.

Projecting the variation

$$\delta g^{ij} = -\nabla_g^i v^j - \nabla_g^j v^i - 2g^{ij}\delta\omega,$$

onto $\bar{\phi}_n$, we find

$$\bar{h}_m(\bar{\chi}_n) = \begin{cases} -\left\langle \bar{\phi}_n^{ij}, \left(\nabla_g^k \bar{v}_m^l + \nabla_g^l \bar{v}_m^k\right)\right\rangle_{\bar{g}}, & \bar{h}_m \equiv \bar{v}_p^i, \\ -2\left\langle \bar{\phi}_n^{ij}, g^{kl}\bar{\omega}_m\right\rangle_{\bar{g}}, & \bar{h}_m \equiv \bar{\omega}_q, \end{cases}$$

The determinant of this matrix can be expressed in terms of a Grassmann integral as

$$\int \prod d\bar{c}_m \prod d\bar{b}_n \; \exp \sum \bar{b}_n \bar{c}_m \, \bar{h}_m(\bar{\chi}_n)$$

$$= \int [dc]_{\bar{g}} [dc^\omega]_{\bar{g}} [db]_{\bar{g}} \; \exp \left\{ - \sum \bar{b}_n \bar{c}_m \, \left\langle \bar{\phi}_n^{ij}, \left(\nabla_g^k \bar{v}_m^l + \nabla_g^l \bar{v}_m^k \right) \right\rangle_{\bar{g}} - 2 \sum \bar{b}_n \bar{c}_m^\omega \, \left\langle \bar{\phi}_n^{ij}, g^{kl} \bar{\omega}_m \right\rangle_{\bar{g}} \right\}$$

$$= \int [dc]_{\bar{g}} [dc^\omega]_{\bar{g}} [db]_{\bar{g}} \; \exp \left\{ \int \sqrt{\bar{g}} \, b_{kl} \left(-\nabla_g^k c^l - \nabla_g^l c^k - 2 \, g^{kl} c^\omega \right) \right\}, \quad (31)$$

by definition of the measure for the ghost fields discussed in section 7, if we define $\bar{b}_n$ by

$$\sum \bar{b}_n \, \bar{\phi}_n^{ij} \bar{g}_{ik} \bar{g}_{jl} = b_{kl},$$

since $\bar{\phi}_n^{ij} \, \bar{g}_{ik} \bar{g}_{jl}$ is a $\bar{g}$-orthonormal basis for the space of fields b_{ij}, and similarly for c and c^ω.

Diffeomorphism invariance, Weyl invariance and $\bar{g}$-independence of the functional measure require the introduction of Pauli-Villars partners v^i, ω and ρ_{ij} for the ghosts c^i, c^ω and b_{ij}. Here, as before, we use for example v^i as a shorthand to denote a set of massive fields whose total statistics is opposite to that of c^i, and whose bosonic elements have to be taken complex-valued to give a non-vanishing mass term. For full details of the construction, we refer back to section 7. The full action for these fields will be stated below. We remark that the bosonic ρ_{ij}, though complex-valued, are symmetric tensors, not hermitian.

We see that a correct application of the Faddeev-Popov procedure gives the following action.

$$S \equiv \frac{1}{2} \int \sqrt{g} \, g^{ij} \, \partial_i X \, \partial_j X + \int \sqrt{\bar{g}} \, b_{ab} \left(-\nabla_g^a c^b - \nabla_g^b c^a - 2 \, g^{ab} \, c^\omega \right)$$
$$- \, i \int \sqrt{\bar{g}} \, B_{ij} \, (g^{ij} - \hat{g}^{ij}) + \cdots ,$$

where the dots indicate Pauli-Villars terms, to be fixed below from the requirement of BRST invariance. The action is of course coordinate-independent, but it is important to note that it has an explicit background-dependence, as embodied in the presence of $\bar{g}$ in both the ghost and the gauge-fixing terms. We shall see that no physical quantity will depend on $\bar{g}$.

This form of the action will be most convenient for discussing BRST-invariance. However, for many calculations it will be more convenient to use a background-independent form of the ghost terms. Such a representation will be derived in the next section, at the cost of obscuring BRST invariance.

The ghosts c^i and c^ω may be identified with the left-invariant one-forms on the gauge group, and gauge invariance of the original action can then be interpreted as invariance under a BRST transformation δ_B which may be identified with the exterior derivative on the gauge group [7]. As an exterior derivative in disguise, the BRST transformation is then automatically nilpotent:

$$\delta_B^2 = 0.$$

The BRST transformations of the ghosts c^i and c^ω are obtained from the Maurer-Cartan structure equations

$$dc^m = -\frac{1}{2}\, C^m_{np}\, c^n c^p$$

that encode the exterior derivatives of a basis c^m of left-invariant one-forms on the group, which depend on the commutation relations of the generators via the structure constants C^m_{np} [7]. In this case the group is generated by vector fields and Weyl transformations, whose commutators can be calculated relatively easily. This gives the transformation of c^i and c^ω. The transformations of b_{ij} and B_{ij} are postulated in a standard way [21, 22]. The transformation of the other fields are obtained by simply lifting the Lie algebra action onto the space of fields. We obtain

$$\begin{aligned}
\delta_B X &= \mathcal{L}_c X = c^i \partial_i X, \\
\delta_B g^{ij} &= \mathcal{L}_c g^{ij} - 2\, g^{ij} c^\omega = -\nabla^i c^j - \nabla^j c^i - 2\, g^{ij} c^\omega, \\
\delta_B c^i &= -c^j \partial_j c^i = -\frac{1}{2}\, \mathcal{L}_c c^i, \\
\delta_B c^\omega &= -c^i \partial_i c^\omega = -\mathcal{L}_c c^\omega, \\
\delta_B b_{ij} &= -i B_{ij}, \\
\delta_B B_{ij} &= 0.
\end{aligned}$$

Here $\mathcal{L}_c$ denotes the Lie derivative, with respect to the vector field $c \equiv c^i \partial_i$, on the appropriate space of tensors. As is obvious from their re-expression as Lie derivatives, the ghost transformations are covariant. But note that the ghosts transform with different coefficients and opposite signs to the other fields.

Invariance of the action follows from the observation that the second and third terms can be written as

$$\int \delta_B \left(\sqrt{\bar{g}}\, b_{ab}\, g^{ab}\right) + i \int \sqrt{\bar{g}}\, B_{ab}\, \hat{g}^{ab}$$

The BRST variation of the first of these vanishes by $\delta_B^2 = 0$, which is satisfied by construction as noted above, while the variation of the second vanishes trivially.

The Pauli-Villars fields are all taken to transform as tensors

$$\begin{aligned}
\delta_B \chi &= \mathcal{L}_c \chi = c^i \partial_i \chi, \\
\delta_B \gamma^{ij} &= \mathcal{L}_c \gamma^{ij} - 2\, \gamma^{ij} c^{\omega}, \\
\delta_B \beta_{ij} &= \mathcal{L}_c \beta_{ij}, \\
\delta_B v^i &= \mathcal{L}_c v^i = [c, v] \\
\delta_B \omega &= \mathcal{L}_c \omega = c^i \partial_i \omega, \\
\delta_B \rho_{ij} &= \mathcal{L}_c \rho_{ij}.
\end{aligned}$$

Note that B and β do not transform in the same way, and neither do the partners c and v, nor c^{ω} and ω. Also, note that γ transforms under conformal transformations in the same way as g.

The Pauli-Villars regularization will require small masses m to be given to the ghost fields, and large masses to their auxiliary Pauli-Villars fields. The form of the mass terms are taken to be

$$m \int \sqrt{g}\, \left(\sqrt{g}\, dx^1 \wedge dx^2\right) (c, c) = m \int \sqrt{g}\, (\sqrt{g}\, \epsilon_{kl})\, c^k c^l$$

and

$$m \int \sqrt{g}\, \left(\sqrt{g}\, \epsilon_{kl}\right) g^{kp} g^{lq} g^{ij}\, b_{ki} b_{lj}$$

respectively. These are diffeomorphism-invariant due to coordinate-invariance of the density $\sqrt{g}\, dx^1 \wedge dx^2$. The full action is then

$$
\begin{aligned}
S \equiv\ & \frac{1}{2}\int \sqrt{g}\, g^{ij}\, \partial_i X\, \partial_j X + \int \sqrt{\bar{g}}\, b_{ab} \left(-\nabla_g^a c^b - \nabla_g^b c^a - 2\, g^{ab}\, c^\omega\right) \\
& - i \int \sqrt{\bar{g}}\, B_{ij}\, (g^{ij} - \hat{g}^{ij}) \\
& + \frac{1}{2}\int \sqrt{g}\left(g^{ij}\, \partial_i \bar{\chi}\, \partial_j \chi + M_\chi^2\, \bar{\chi}\chi\right) + \frac{1}{2}\int \sqrt{g}\left\{\bar{\rho}_{ab}\left(-\nabla_g^a v^b - \nabla_g^b v^a - 2\, g^{ab}\, \omega\right) + \text{c.c.}\right\} \\
& + M_v \int \sqrt{g}\, (\sqrt{g}\, \epsilon_{kl})\, \bar{v}^k v^l + M_b \int \sqrt{g}\, \left(\sqrt{g}\, \epsilon_{kl}\right) g^{kp} g^{lq} g^{ij}\, \bar{\rho}_{ki} \rho_{lj} \\
& - \frac{i}{2}\int \sqrt{g}\left(\bar{\beta}_{ij}\, \gamma^{ij} + \text{c.c.}\right).
\end{aligned}
\tag{32}
$$

It is important to note that the Pauli-Villars terms are taken to be diffeomorphism invariant, independent of $\bar{g}$, in contrast with the original terms. As a result, they are invariant under the component of the BRST transformation parameterized by the vector c^i, since this simply acts as an infinitesimal diffeomorphism on the Pauli-Villars terms. Notice, though, that the mass terms break the Weyl component of the BRST transformation generated by c^ω. This will be origin of the conformal anomaly in this approach.

In the above action, we have assumed the shorthand of writing only one Pauli-Villars partner for each ordinary field. In practice we usually need a few, enough to satisfy the required conditions on the statistics and the masses [1].

An important property of the current construction is that the measure is BRST-invariant. This is due to certain perhaps unexpected cancellations of the BRST variations of different factors. First, note that trivially

$$\delta_B[dB] = 0.$$

Also

$$\delta_B\left([db]\,[dB]\right) = 0,$$

by the elementary property $db_n \wedge db_n = 0$ of the exterior product. Also, per the discussion of section 7, we have

$$\delta_B\left([dX]\,[d\chi]\right) = \delta_B\left([dg]\,[d\gamma]\right) = \delta_B\left([dc^\omega]\,[d\omega]\right) = 0,$$

since in each case the two factors transform with opposite signs due to opposite statistics. The same is true for the following pair of factors, even though they are not Pauli-Villars partners:

$$\delta_B\left([\rho]\,[\beta]\right) = 0.$$

Finally, a slightly nontrivial calculation shows that

$$\delta_B\left([dc]\,[dv]\right) = 0.$$

Even though the BRST transformations of c^i and v^i differ by a factor of 2, this is compensated by the fact that $\delta_B c^i$ is quadratic in c^i while $\delta_B v^i$ is linear.

We now consider insertions. Before gauge fixing, each insertion contributes

$$\int d^2x\,\sqrt{g}\,V(x)$$

to the path integral. This is diffeomorphism-invariant, though not in general Weyl-invariant; requiring the absence of the corresponding quantum anomaly will determine the physical state condition on V. After using diffeomorphism invariance of the full path integral to fix the metric to a fixed $\hat{g}$ up to a Weyl transformation, a finite set of diffeomorphisms, Weyl related to global conformal transformations, may remain that can be used to fix the positions $\hat{x}$ of a finite set of insertions. For each of these, the gauge fixing function may be chosen as

$$\delta(x-\hat{x})$$

and the Faddeev-Popov determinant contribution may be written as [12]

$$\int d^2\eta\, e^{\eta_i c^i(x)},$$

where η_i is an anti-commuting covector. A simple calculation then gives the following contribution for a fixed insertion

$$\int d^2x\,\sqrt{g}\,V(x)\,\delta(x-\hat{x})\int d^2\eta\, e^{\eta_i c^i(x)} = \sqrt{g(\hat{x})}\,c^1(\hat{x})\,c^2(\hat{x})\,V(\hat{x}). \qquad (33)$$

As an aside, it is worth noting that this can be rewritten in terms of a measure that is both diffeomorphism- and Weyl invariant as follows:

$$\int d^2x\sqrt{g}\,V(x)\,\delta_g(x-\hat{x})\int \frac{d^2\eta_i}{\sqrt{g}}\,e^{\eta^i g_{ij}\,c^j(x)},$$

which shifts all the Weyl variance to the integrand.

11 Covariantizing the ghost term

As discussed in the previous section, the full action (32) is BRST invariant except for the Pauli-Villars mass terms that will contribute the conformal anomaly. However, notice that the ghost kinetic term depends on the background metric $\bar{g}$. For some calculations, it is more convenient to replace this term with a background-independent form. In this section we will show that one can do this while making a corresponding change in the β-γ term, at the cost of obscuring the manifest BRST invariance of the action.

We first consider the partition function without insertions. Let $\bar{\phi}_m$ be an orthonormal basis for the space of 2-tensors with respect to the $\bar{g}$-inner product and ϕ_m a basis with respect to the g-inner product. Similarly, let $\bar{\varphi}_n$ and φ_n be orthonormal bases for the space $V^1 \oplus V^0$, where v^1 denotes the space of vector and v^0 the space of scalar fields, with respect to the $\bar{g}$-inner product and the g-inner product respectively. Also, denote

$$i_g(v \oplus \omega) \equiv \nabla_g^i v^j + \nabla_g^j v^i - 2\, g^{ij} \omega.$$

The relevant factors in (32) can be represented as

$$
\begin{aligned}
&\int [db]_{\bar g}\,[dc]_{\bar g}\,[d\rho]_{\bar g}\,[dv]_{\bar g}\,[dg]_{\bar g}\,[d\gamma]_{\bar g}\,[dB]_{\bar g}\,[d\beta]_{\bar g}\, e^{\langle b|i_g c\rangle_{\bar g}} e^{\frac{1}{2}\left(\langle\beta|\gamma\rangle_g + \text{c.c.}\right)}\ldots \\
&\quad = \int [db]_{\bar g}\,[dc]_{\bar g}\,[d\rho]_{\bar g}\,[dv]_{\bar g}\,[dg]_g\,[d\gamma]_g\,[dB]_g\,[d\beta]_g\, e^{\langle b|i_g c\rangle_{\bar g}} e^{\langle\beta|\gamma\rangle_g}\ldots \\
&\quad = \int [db]_{\bar g}\,[dc]_{\bar g}\,[d\rho]_{\bar g}\,[dv]_{\bar g}\,[dg]_g\,[dB]_g\, e^{\langle b|i_g c\rangle_{\bar g}}\ldots \\
&\quad = \int [d\rho]_{\bar g}\,[dv]_{\bar g}\,[dg]_g\,[dB]_g\ \det\left\langle\bar\phi_m\middle|i_g\bar\varphi_n\right\rangle_{\bar g}\cdots \\
&\quad = \int [d\rho]_{\bar g}\,[dv]_{\bar g}\,[dg]_g\,[dB]_g\ \det\left\langle\bar\phi_m\middle|\phi_n\right\rangle_{\bar g}\ \det\left\langle\phi_n\middle|i_g\varphi_p\right\rangle_g\ \det\left\langle\varphi_p\middle|\bar\varphi_q\right\rangle_g\cdots \\
&\quad = \int [d\rho]_{\bar g}\,[dv]_g\,[dg]_g\,[dB]_g\ \det\left\langle\bar\phi_m\middle|\phi_n\right\rangle_{\bar g}\ \det\left\langle\phi_n\middle|i_g\varphi_p\right\rangle_g\ \cdots \\
&\quad = \int [d\rho]_g\,[dv]_g\,[dg]_g\,[dB]_g\ \det\left\langle\phi_p\middle|\bar\phi_m\right\rangle_{\bar g}\ \det\left\langle\bar\phi_m\middle|\phi_n\right\rangle_{\bar g}\ \det\left\langle\phi_n\middle|i_g\varphi_p\right\rangle_g\ \cdots \\
&\quad = \int [d\rho]_g\,[dv]_g\,[dg]_g\,[dB]_g\ \det\left\langle\phi_p\middle|\phi_n\right\rangle_{\bar g}\ \det\left\langle\phi_n\middle|i_g\varphi_p\right\rangle_g\ \cdots \\
&\quad = \int [db]_g\,[dc]_g\,[d\rho]_g\,[dv]_g\,[dg]_g\,[d\gamma]_g\,[dB]_g\,[d\beta]_g\, e^{\frac{1}{2}\left(\langle\beta|\gamma\rangle_{\bar g} + \text{c.c.}\right)} e^{\langle b|i_g c\rangle_g}\ldots \\
&\quad = \int [db]_{\bar g}\,[dc]_{\bar g}\,[d\rho]_{\bar g}\,[dv]_{\bar g}\,[dg]_{\bar g}\,[d\gamma]_{\bar g}\,[dB]_{\bar g}\,[d\beta]_{\bar g}\, e^{\frac{1}{2}\left(\langle\beta|\gamma\rangle_{\bar g} + \text{c.c.}\right)} e^{\langle b|i_g c\rangle_g}\ldots
\end{aligned}
$$

where we have treated various infinite-dimensional determinants informally. In the second line we used $\bar g$-invariance of the combined matter-Pauli-Villars measures. In the third line we integrated over β and γ to obtain 1. In the sixth line we used

$$[dv]_g = [dv]_{\bar g}\ \det\left\langle\varphi_p\middle|\bar\varphi_q\right\rangle_g\,,$$

and then

$$[d\rho]_{\bar g} = [d\rho]_g\ \det\left\langle\phi_p\middle|\bar\phi_m\right\rangle_{\bar g}\,.$$

We obtain the following action:

$$
\begin{aligned}
S \equiv\ & \frac{1}{2}\int \sqrt{g}\, g^{ij}\, \partial_i X\, \partial_j X + \int \sqrt{g}\, b_{ab}\left(-\nabla_g^a c^b - \nabla_g^b c^a - 2\, g^{ab}\, c^\omega\right) \\
& - i\int \sqrt{\bar{g}}\, B_{ij}\, (g^{ij} - \hat{g}^{ij}) \\
& + \frac{1}{2}\int \sqrt{g}\left(g^{ij}\, \partial_i\bar{\chi}\, \partial_j\chi + M_\chi^2\, \bar{\chi}\chi\right) + \frac{1}{2}\int \sqrt{g}\left\{\bar{\rho}_{ab}\left(-\nabla_g^a v^b - \nabla_g^b v^a - 2\, g^{ab}\, \omega\right) + \text{c.c.}\right\} \\
& + M_v \int \sqrt{g}\, (\sqrt{g}\, \epsilon_{kl})\, \bar{v}^k v^l + M_b \int \sqrt{g}\ (\sqrt{g}\, \epsilon_{kl})\, g^{kp} g^{lq} g^{ij}\, \bar{\rho}_{ki}\rho_{lj} \\
& - \frac{i}{2}\int \sqrt{\bar{g}}\left(\bar{\beta}_{ij}\, \gamma^{ij} + \text{c.c.}\right). \qquad (34)
\end{aligned}
$$

Compared with the original action (32), the b-c term is now $\bar{g}$-independent, while the β-γ term now depends on $\bar{g}$.

We now consider the insertions. These are, in fact, unaffected by the above manipulations, since

$$
\int [dc]_{\bar{g}}\, [dv]_{\bar{g}} \int d^2\eta\, e^{\eta_i c^i(x)} = \int [dc]_g\, [dv]_g \int d^2\eta\, e^{\eta_i c^i(x)}
$$

or

$$
\int [dc]_{\bar{g}}\, [dv]_{\bar{g}} \int d^2\eta\, e^{\eta_i \sum \bar{\varphi}_n^i \bar{c}_n} = \int [dc]_g\, [dv]_g \int d^2\eta\, e^{\eta_i \sum \varphi_n^i c_n}
$$

The insertions are therefore still given by (33).

12 BRST anomalies

It is straightforward to check, from the BRST-invariant form (32) of the action, that

$$
\delta_B S = -\frac{1}{4\pi}\int d^2x\, \sqrt{g}\, (-2\, c^\omega)\, T^i_{\ i}(\chi, \rho, v, \omega). \qquad (35)
$$

Only the contributions to $T^i_{\ i}$ from the Pauli-Villars mass terms contribute here, since the c^ω-variance of the ghost term cancels that of the (B, g) gauge fixing term. Also note that when using the form (32), the (β, γ) term is conformally invariant and therefore does not contribute a c^ω variance to $\delta_B S$.

We may obtain a current associated with δ_B by a standard procedure as follows. Write the change of variables formula for the path integral

$$\int [d\phi]' \, e^{-\tilde{S}(\phi')} = \int [d\phi] \, e^{-\tilde{S}(\phi)}$$

where ϕ stands for all the fields in the theory, and where we take the change of variables

$$\phi'(x) = \phi(x) + \epsilon(x) \, \delta_B \phi(x),$$

where $\epsilon(x)$ denotes a Grassmann-valued function. Since this is a symmetry of the massless action for ϵ constant, the variation of this part of the action is proportional to derivatives of ϵ which, after a partial integration, would give rise to a conserved current [12, 1] in the absence of additional mass terms. Here, the presence of the Pauli-Villars mass terms will give additional contributions.

We note that, as discussed already, the measure is invariant under δ_B. With this in mind, only the variation of the action contributes in the above formula. We start by taking the non-covariantized version (32) of the action, and we obtain

$$0 = \frac{1}{4\pi} \int d^2x \left\{ (2 \, \nabla_g^i \epsilon) \, \left\langle \left(\sqrt{g} \, c^j \, T_{ij}(X, \chi, \rho, v, \omega) + \sqrt{\bar{g}} \, b_{ik} \, c^j \partial_j c^k \right) \cdots \right\rangle \right.$$
$$\left. + \, \epsilon \left\langle \sqrt{g} \, (-2 \, c^\omega) \, T^i_i(\chi, \rho, v, \omega) \cdots \right\rangle \right\} \tag{36}$$

where the dots denote possible additional insertions outside the support of $\epsilon(x)$. To derive this formula, note that for the fields $(X, \chi, \rho, v, \beta, \gamma, \omega)$, the transformation under the c^i-indexed part of δ_B is identical to an infinitesimal diffeomorphism parameterized by c^i, and we obtain the energy-momentum tensor $T_{ij}(X, \chi, \rho, v)$ in the usual way. Note also that the (β, γ) contribution is zero, since this term is both diffeomorphism and Weyl invariant and has no derivatives, so that no ϵ-derivatives are contributed. Next, note that terms proportional to $\epsilon(x)$ cancel between the (b, c) and (B, g) terms as in the case of constant ϵ, and only the terms containing derivatives of ϵ in the variation of the ghost terms remain.

Note that, by the construction in section 10, changing $\bar{g}^{ij}$ at worst changes the overall normalization of the Haar measure. As a result, the path integral is independent, up to a possible overall gauge slice-independent factor, of the background metric $\bar{g}^{ij}$, as long as any further ghost insertions are in the specific format (33) required for fixed vertex operators.

However, the above expression contains an explicit instance of $\bar{g}$ that is not in an overall normalization position, in seeming contradiction with this statement. This is resolved by noting that from the (b, c) term in the action it is easy to see that propagators involving the combination $\sqrt{\bar{g}}\, b_{ij}$ are independent of $\bar{g}$.

This remark also explains why the above expression will be finite due to cancellation of self-contraction divergences between fields and their Pauli-Villars partners, as expected for our covariant regularization. Indeed even though the (ρ, v) term has no $\bar{g}$-dependence, the combination $\sqrt{\bar{g}}\, \rho_{ij}$ will have the same propagator as $\sqrt{\bar{g}}\, b_{ij}$.

Note that we are free to choose $\bar{g}^{ij} = \hat{g}^{ij}$ for a fixed choice $\hat{g}^{ij}$ of gauge slice. However, slice independence of the path integral is only valid as long as we vary $\hat{g}$ independently of $\bar{g}$. Otherwise the Haar normalization factor would become dependent on $\hat{g}$, and gauge slice independence of the path integral would be broken.

To avoid this issue, and to avoid dealing with two different metrics in the resulting conservation law, it is worth comparing the above identity with the one we would have obtained if we had started from the covariantized action (34) instead. It would be

$$
\begin{aligned}
0 = \frac{1}{4\pi} \int d^2x \Bigg\{ & (2\, \nabla_g^i \epsilon)\, \left\langle \left(\sqrt{g}\, c^j\, T_{ij}(X, \chi, \rho, v, \omega) + \sqrt{g}\, b_{ik}\, c^j \partial_j c^k \right) \cdots \right\rangle \\
& + \epsilon \Bigg\langle (-2\, c^\omega) \Bigg(\sqrt{g}\, T^i_{\ i}(\chi, \rho, v, \omega) \\
& \qquad + \sqrt{g}\, b_{ab} \left(-\nabla_g^a c^b - \nabla_g^b c^a - 2\, g^{ab}\, c^\omega \right) + \frac{i}{2} \sqrt{\bar{g}}\, \left(\bar{\beta}_{ij}\, \gamma^{ij} + \text{c.c.} \right) \Bigg) \cdots \Bigg\rangle \Bigg\},
\end{aligned}
$$

and since the actions were argued to be equivalent, this identity should be equivalent to the first identity (36) above as long as any further ghost insertions are in the specific format (33).

To see that this is true, note that the ghost term in the first line now depends on $\sqrt{g}$, but now so does the corresponding term in the action (34), so that propagators involving $\sqrt{g}\, b_{ij}$ will be the same as propagators involving $\sqrt{\bar{g}}\, b_{ij}$ in the first version above. The new terms on the last line will compensate each other. In particular, from the action we can read off that the contraction of $\sqrt{g}\, b_{ab}$ with $-\nabla_g^a c^b - \nabla_g^b c^a - 2\, g^{ab}\, c^\omega$ is the same as the contraction of $-\frac{i}{2}\,\beta_{ij}$ with γ^{ij}. As a result, self-contractions on the last line cancel. Furthermore, if $\cdots$ contains vertices with accompanying c-ghost factors, these may contribute contractions with b_{ab} in the first term, but the result is then proportional to $-\nabla_g^a c^b - \nabla_g^b c^a - 2\,g^{ab}\,c^\omega$, which vanishes by the equations of motion in the expectation value under the stated assumption on the content of $\cdots$. We also assume that $\cdots$ contains no (β,γ) dependent insertions. As a result of all these arguments, both terms on the last line may be dropped completely, and we obtain a nice identity without explicit $\bar{g}$-dependence.

$$0 = \frac{1}{4\pi}\int d^2x \left\{ (2\,\nabla^i\epsilon)\,\left\langle\left(\sqrt{g}\,c^j\,T_{ij}(X,\chi,\rho,v,\omega) + \sqrt{g}\,b_{ik}\,c^j\partial_j c^k\right)\cdots\right\rangle \right.$$
$$\left. + \,\epsilon\left\langle(-2\,c^\omega)\;\sqrt{g}\,T^i_{\;i}(\chi,\rho,v,\omega)\;\cdots\right\rangle\right\},$$

Performing the integral over g, which applies the gauge-fixing delta function fixing $g=\hat{g}$, we get

$$0 = \int d^2x\,\sqrt{\hat{g}}\,\epsilon(x)\;\left\langle\left\{-\nabla^i\left(c^jT_{ij}(X,\chi,\rho,v,\omega) + b_{ik}\,c^j\partial_j c^k\right) - c^\omega T^i_{\;i}(\chi,\rho,v,\omega)\right\}\cdots\right\rangle_{\hat{g}}.$$

Since $\epsilon(x)$ is arbitrary, we find the covariant conservation law

$$\left\langle\nabla^i j_i\cdots\right\rangle_{\hat{g}} = -\left\langle c^\omega\,T^i_{\;i}(X,\chi,\rho,v,\omega)\cdots\right\rangle_{\hat{g}}$$

as long as no other insertion is at the point in question. Here the BRST current is

$$j_i \equiv c^jT_{ij}(X,\chi,\rho,v,\omega) + b_{ik}\,c^j\partial_j c^k$$

Remembering that $c^j\partial_j c^k = \frac{1}{2}\,\mathcal{L}_c c^k$, we note that $j_i\,dx^i$ is indeed a coordinate-invariant, true tensor quantity. Also note that the anomalous quantity $T^i_{\;i}(\chi,\rho,v,\omega)$ on the right hand side does not include contributions from (B,g,β,γ), even though the corresponding terms in (34) are not Weyl invariant.

When no other insertion coincides with $T^i_{\;i}$, we may replace [1]

$$T^i_{\;i} \to -\frac{c}{12}\,R,$$

where R is the curvature of $\hat{g}$ and c is the total central charge, and we get

$$\left\langle\nabla^i j_i\cdots\right\rangle = \frac{c}{12}\,R\,\langle c^\omega\cdots\rangle\,.$$

Finally, c will be calculated in section 13 and shown to be zero in dimension $d = 26$, in which case we indeed get a non-anomalous covariant conservation law.

We may also use the equation of motion, valid in expectation values, to replace

$$c^\omega \to -\frac{1}{2}\left(\nabla^i c^j + \nabla^j c^i\right)$$

on the right hand side.

It is essential to understand that, just like the energy-momentum tensor in the covariant regularization [1], this current is *finite*. Divergences arising from self-contractions of the second term are exactly canceled by self-contractions of the Pauli-Villars contribution $c^j T_{ij}(\rho, v)$. As a result, no covariance-spoiling renormalization is needed, in contrast with the usual operator formalism [12],.

Since the Pauli-Villars regularization is coordinate-independent, and provides finite insertions without the need for further renormalizations [1], the BRST current $j_i\, dx^i$ is by construction a true coordinate-independent tensor quantity. In this the current formalism differs from the operator formalism, where the renormalizations depend on a coordinate choice, leading to an anomalous transformation law for the current [12]. The j constructed here does not transform anomalously but instead contains extra contributions. To see this clearly, take $\hat{g}$ to be the flat metric, so that we may write

$$\begin{aligned} j_z = {} & c^z T_{zz}(X, \chi, \rho, v, \omega) + b_{zz}\, c^z \partial_z c^z && (37)\\ & + c^{\bar z} T_{z\bar z}(\chi, \rho, v, \omega) + b_{z\bar z}\, c^z \partial_z c^{\bar z} + b_{z\bar z}\, c^{\bar z} \partial_{\bar z} c^{\bar z} + b_{zz}\, c^{\bar z} \partial_{\bar z} c^z. && (38) \end{aligned}$$

The terms on the second line do not vanish in our covariant regularization, since they may contribute contact terms to expectation values. It is obvious that dropping some of these terms, as is done in the usual operator formalism, would lead to a non-covariant, non-tensor object.

We may follow a similar procedure to obtain Ward identities involving insertions, where the current acts as generator for the δ_B. Concretely, if $\mathcal{O}(\hat{x})$ is an isolated insertion, the same argument, using an $\epsilon(x)$ that is nonzero only in a neighbourhood of $\hat{x}$ that contains no other insertions, gives the Ward identity

$$0 = \epsilon(\hat{x})\, \langle \delta_B \mathcal{O} \cdots \rangle_{\hat{g}} - \frac{1}{2\pi} \int d^2x\, \sqrt{\hat{g}}\, \epsilon(x)\ \left\langle \mathcal{O}(\hat{x}) \left\{ \nabla^i j_i + c^\omega T^i_{\ i}(\chi, \rho, v, \omega) \right\} \cdots \right\rangle_{\hat{g}} \qquad (39)$$

If we take $\epsilon(x) = \epsilon$ on a neighbourhood of $\hat{x}$ and $\epsilon \to 0$ outside this neighbourhood, the first term is then a total derivative, which can be integrated to give a BRST charge. As noted above in (38), this charge will contain terms proportional to boundary integrals of

$$T_{z\bar{z}}(X, \chi, \rho, v, \omega) + b_{z\bar{z}}\, c^z \partial_z c^{\bar{z}} + b_{z\bar{z}}\, c^{\bar{z}} \partial_{\bar{z}} c^{\bar{z}} + b_{zz}\, c^{\bar{z}} \partial_{\bar{z}} c^z$$

that are not there in the usual operator formalism.

However, since these extra contributions may only contribute contact terms, and since the insertion in (39) does not touch the boundary, we may drop them to obtain the usual holomorphic BRST charges. Note that the Pauli-Villars contributions in

$$c^z T_{zz}(X, \chi, \rho, v, \omega)$$

cannot be dropped, even though they can only give rise to contact terms when contracted with the insertion, since they also contribute self-contractions that cancel divergences arising from self-contractions of the matter and the ghost terms [1].

In (39), it is important to note that the second term on the right hand side is not a total derivative, and cannot be directly converted into a surface integral. Since it can touch the insertion, we cannot simplify it as we did in the case of current conservation above. Due to the presence of this term, the formula (39) generates the *classical*, non-anomalous BRST transformation δ_B.

This is indeed the expected behaviour of a coordinate-invariant, Pauli-Villars based regularization, as we have seen also in the case of the energy-momentum tensor Ward identities in section 5. As there, this Ward identity can be related to the usual operator formalism identity [12] by non-covariant operator redefinitions, which would be as unnatural in the present context as they were in section 5. Possible anomalies, leading to the physical state conditions, present themselves in a different way, as we shall see below.

At first glance it may seem as if the above Ward identity cannot possibly be correct. Since j_i contains no instances of the field B_{ij} conjugate to g^{ij}, it would appear that the formula (39) cannot be used to generate transformations of $\mathcal{O} = g^{ij}$. However, one should remember that we are deriving identities that are true inside expectation values. For example, taking $\mathcal{O} = g^{ij}$, the right hand side of (39) vanishes, while the left hand side is just the expectation value of $\delta_B g^{ij}$, so that we obtain

$$\left\langle -\nabla^i c^j - \nabla^j c^i - 2\, c^\omega \hat{g}^{ij} \cdots \right\rangle_{\hat{g}} = 0, \tag{40}$$

which simply states that the expectation values satisfy the classical equations of motion.

The physical state condition may be recast as follows in the BRST language. We restate the requirement, considered in section 8, that

$$\langle V(\hat{x}) \cdots \rangle$$

be independent of the choice of gauge fixing function indexed by $\hat{g}$.

$$\begin{aligned}
0 &= -i \int d^2x \sqrt{\overline{g}} \left\langle \tilde{V}(\hat{x})\, B_{ij}(x) \cdots \right\rangle \delta\hat{g}^{ij}(x) \\
&= \int d^2x \sqrt{\overline{g}} \left\langle \tilde{V}(\hat{x})\, \delta_B b_{ij}(x) \cdots \right\rangle \delta\hat{g}^{ij}(x) \\
&= \int d^2x \sqrt{\overline{g}} \int d\mu\, \tilde{V}(\hat{x})\, e^{-S} \left(\delta_B b_{ij}(x)\right) \delta\hat{g}^{ij}(x) \cdots \\
&= - \int d\mu \left((\delta_B \tilde{V})(\hat{x})\, e^{-S} - \int d\mu\, V(\hat{x})\, (\delta_B S)\, e^{-S} \right) \int d^2x \sqrt{\overline{g}}\, b_{ij}(x)\, \delta\hat{g}^{ij}(x) \cdots + \cdots \\
&= - \left\langle \Big(\delta_B - \delta_B S \Big) \tilde{V}(\hat{x}) \int d^2x \sqrt{\overline{g}}\, b_{ij}(x)\, \delta\hat{g}^{ij}(x) \cdots \right\rangle + \cdots,
\end{aligned}$$

where we may use either action (32) or (34). Again we have used that the measure is BRST-invariant, or

$$\int d\mu\, \delta_B(\cdots) = 0.$$

The fact that we shall obtain the correct physical state conditions constitutes an independent check that this is indeed correct.

It is now obvious where the quantum condition differs from the classical one. Indeed, as noted in (35), the Pauli-Villars regularization contributes mass terms to S, so that

$$\delta_B S = -\frac{1}{4\pi} \int d^2x \sqrt{g}\, (-2\, c^\omega)\, T^i_{\ i}(\chi, \rho, v, \omega) \neq 0.$$

The trace of the energy-momentum tensor contributes contact terms in the limit of infinite Pauli-Villars masses. For example, for the tachyon

$$\tilde{V} = \sqrt{g}\,\epsilon_{ij}\,c^i c^j\,e^{ik\tilde{X}},$$

we obtain, in conformal coordinates,

$$\delta_B\,(\sqrt{g}\,c\bar{c}) = 2\,c^\omega\,(\sqrt{g}\,c\bar{c}),$$

where terms containing ∂c and $\bar{\partial}\bar{c}$ have canceled between $\delta_B\sqrt{g}$ and $\delta_B(c\bar{c})$, so that

$$0 = \left\langle \left(2\,c^\omega(z)\,\tilde{V}(z) + \frac{1}{4\pi}\int d^2w\,(-4\,c^\omega(w))\,(T_{w\bar{w}} + T_{\bar{w}w})\,\tilde{V}(z)\right) \int d^2u\,b_{ij}(u)\,\delta\hat{g}^{ij}(u)\cdots \right\rangle.$$

Choosing $\delta\hat{g}^{ij} = -2\,\hat{g}^{ij}\,\delta\omega$, and then contracting

$$c^\omega(z)\,b_i{}^i(u) \sim \delta^2(z,u)$$

gives

$$0 = \left\langle \left(2\,\delta\omega(z)\,\tilde{V}(z) + \frac{1}{4\pi}\int d^2w\,(-4\,\delta\omega(w))\,(T_{w\bar{w}} + T_{\bar{w}w})\,\tilde{V}(z)\right) \cdots \right\rangle,$$

which is exactly the same condition as the one obtained for the tachyon in section 8.

The requirement of gauge slice invariance of the partition function with no insertions can be restated by taking $\tilde{V} = 1$ in the above argument to get

$$0 = \int d^2x\,\sqrt{\hat{g}}\;\langle \delta_B b_{ij}\cdots\rangle\,\delta\hat{g}^{ij}. \tag{41}$$

If this is violated, the theory has a conformal anomaly.

Such an anomaly can also be restated a non-invariance of the effective action under a quantum version of the BRST transformation to be defined below. In the Lagrangian formalism considered here, this will be the most natural statement of the anomaly, usually encoded in the operator formalism in terms of $Q^2 \neq 0$. Note that our transformation δ_B acts on the classical fields involved in the path integral and satisfies

$$\delta_B^2 = 0$$

by construction. Given this property of δ_B, then in accordance with the discussion of this issue in [21], any possible anomaly will be reflected in a failure of invariance of the effective action Γ under the quantum version of the BRST transformation δ_B which, following the analysis of [22], is expressed as

$$\int d^2x \sqrt{\overline{g}} \, \langle \delta_B \Phi_i \rangle_J \, \frac{\partial \Gamma(\phi)}{\partial \phi_i} = 0,$$

where Φ_i ranges over all the fields in the action, and J denotes a set of sources for the fields Φ_i. The values of the sources J are fixed by the requirement that $\phi_i \equiv \langle \Phi_i \rangle_J$.

It is now convenient to use the ghost-covariantized form (34) of the action, so that we can use manipulations similar to those performed in section 8. The only possibly non-vanishing contribution to the above integral is then given by the term

$$\begin{aligned}
\int d^2x \sqrt{\overline{g}} \, \langle \delta_B \, b_{ij} \rangle \frac{\partial \Gamma}{\partial b_{ij}} &= \int d^2x \sqrt{\overline{g}} \, \langle B_{ij} \rangle \frac{\partial \Gamma}{\partial b_{ij}} \\
&= -\frac{1}{4\pi} \int d^2x \, \langle \sqrt{g} \, T_{ij}(X, \chi, b, c, \rho, v, \omega) \rangle \frac{\partial \Gamma}{\partial b_{ij}} \\
&= -\frac{1}{4\pi} \int d^2x \sqrt{\hat{g}} \, \langle T_{ij} \rangle_{\hat{g}} \frac{\partial \Gamma}{\partial b_{ij}} \qquad (42) \\
&= \frac{c}{48\pi} \int d^2x \sqrt{\hat{g}} \, \langle T^i_{\ i} \rangle_{\hat{g}} \, \frac{\partial \Gamma}{\partial b_\omega} + \cdots \\
&= \frac{c}{48\pi} \int d^2x \sqrt{\hat{g}} \, R_{\hat{g}} \frac{\partial \Gamma}{\partial b_\omega} + \cdots , \qquad (43)
\end{aligned}$$

where $b_\omega \equiv b_i^{\ i}$, where $R_{\hat{g}}$ denotes the curvature, and where c measures the combined anomaly of the matter and ghost energy-momentum tensor of the action S. Since we are using the action (34), there is no β-γ contribution.

Although this always vanishes for a flat world sheet, remember that the effective action is a functional of all the fields in the original action, including all configurations of g_{ij}. It is not sufficient for its variation to vanish only on a set of critical points in the space of g_{ij}. Rather, the variation of the effective action must vanish on its entire domain, which then implies that c must be zero in a consistent gauge-invariant quantization.

We may simply relate the above BRST invariance condition to the previous condition of gauge slice invariance. Identifying

$$\frac{\partial\Gamma}{\partial b_{ij}} \to \delta\hat{g}^{ij},$$

in other words, calling the external source for b_{ij}, needed to define the effective action, by the name $\delta\hat{g}^{ij}$, we see that this BRST-invariance condition is equivalent to the condition (41) for gauge slice invariance of the path integral.

We may also relate the above invariance condition to the anti-bracket formalism [22] as follows. Simply notice that, in our gauge-fixed action, $\hat{g}^{ij}$ is the external field coupling to the variation $\delta_B b_{ij} = B_{ij}$. Considering the effective action $\tilde{\Gamma}$ as a functional also of $\hat{g}_{ij}$, the above equation (43) becomes

$$(\tilde{\Gamma},\tilde{\Gamma}) \equiv \int d^2x\, \frac{\partial\tilde{\Gamma}}{\partial\hat{g}^{ij}}\, \frac{\partial\tilde{\Gamma}}{\partial b_{ij}} = \frac{c}{48\pi}\int d^2x\, \sqrt{\hat{g}}\, R_{\hat{g}}\, \frac{\partial\tilde{\Gamma}}{\partial b_\omega}.$$

The condition that this should vanish is called the Zinn-Justin equation.

13 D=26

In [1], it was shown that each matter field X, together with its Pauli-Villars partners χ_i, contribute $c = 1$ to the anomaly. In this section we show that, in the current formalism, the ghosts, together with their Pauli-Villars partners, contribute $c = -26$ to the anomaly.

As in [1], the anomaly can be read off from the coefficient multiplying the contact term

$$\langle T_{z\bar{z}} T_{w\bar{w}}\rangle\,,$$

which we will now calculate for the ghosts and their partners. Before we do so, starting from the action (34), we change variables

$$c^\omega \to c^\omega - \frac{1}{2}\nabla_i c^i,$$
$$\omega \to \omega - \frac{1}{2}\nabla_i v^i,$$

which has trivial Jacobian in the path integral for the same reason that

$$(dx + \lambda\, dy)\wedge dy = dx\wedge dy,$$

and we obtain a ghost action of the form

$$S_{\mathrm{gh}} = \int \sqrt{g}\, b_{ab} \left(-\nabla^a c^b - \nabla^b c^a + g^{ab} \nabla_i c^i \right) - 2 \int \sqrt{g}\, b_a{}^a\, c^\omega$$
$$+ PV.$$

This action will be simpler than the original in conformal coordinates.

First, we note that the factor

$$\int [db_a]\, [dc^\omega]\, [d\rho_a]\, [d\omega]\, \exp \left(2 \int \sqrt{\hat{g}}\, b_a{}^a\, c^\omega + 2 \int \sqrt{\hat{g}}\, \rho_a{}^a\, \omega \right)$$

in the path integral is independent of $\hat{g}$. This is trivially shown by differentiating with respect to $\sqrt{\hat{g}}$ and using

$$\langle b_a{}^a\, c^\omega \rangle = - \langle \rho_a{}^a\, \omega \rangle \,.$$

As a result, these terms do not contribute to the anomaly and may be ignored. The remaining terms may be expressed in holomorphic coordinates as

$$S = \frac{1}{2} \int \Big\{ - b_{zz}\, (2\bar{\partial})\, c^z - b_{\bar{z}\bar{z}}\, (2\partial)\, c^{\bar{z}}$$
$$- c^z\, (2\bar{\partial})\, b_{zz} - c^{\bar{z}}\, (2\partial)\, b_{z\bar{z}}$$
$$+ e^{-2\omega}\, \tilde{m}\, (b_{zz} b_{\bar{z}\bar{z}} - b_{\bar{z}\bar{z}} b_{zz}) - e^{4\omega}\, m\, (c^z c^{\bar{z}} - c^{\bar{z}} c^z) \Big\} + PV,$$

where we have given small mass terms to the ghosts, to be taken to zero in the end, of the same form of the corresponding Pauli-Villars mass terms in (34). All dependence on $b_{z\bar{z}}$ cancels. For now we have kept the Weyl factor ω defined by $g_{ij} = e^{2\omega}\, \delta_{ij}$ nonzero, so that from

$$\delta_\omega S \equiv \frac{1}{4\pi} \int \sqrt{g}\, (2\delta\omega)\, T^i_{\ i},$$

we may read off

$$T_{z\bar{z}} = 2\pi \left\{ -\frac{1}{2}\, \tilde{m}\, b\bar{b} + m\, c\bar{c} \right\}$$

at $\omega = 0$. Here we abbreviated $b \equiv b_{zz}$, $\bar{b} \equiv b_{\bar{z}\bar{z}}$.

We may read off the propagator of $(b, \bar{b}, c, \bar{c})$ from the action. At the point of interest $\omega = 0$, it is given by

$$\frac{1}{4\,\partial\bar{\partial} + \tilde{m}m} \begin{pmatrix} 0 & m & 2\partial & 0 \\ -m & 0 & 0 & 2\bar{\partial} \\ 2\partial & 0 & 0 & -\tilde{m} \\ 0 & 2\bar{\partial} & \tilde{m} & 0 \end{pmatrix}$$

We would like to calculate the ghost contribution

$$\begin{aligned} \langle T_{z\bar{z}} T_{w\bar{w}} \rangle = (2\pi)^2 \Big\{ & \frac{1}{4}\,\tilde{m}^2 \left\langle b(z)\,\bar{b}(w) \right\rangle \left\langle \bar{b}(z)\,b(w) \right\rangle \\ & + m\tilde{m} \left\langle b(z)\,c(w) \right\rangle \left\langle \bar{b}(z)\,\bar{c}(w) \right\rangle \\ & + m^2 \left\langle c(z)\,\bar{c}(w) \right\rangle \left\langle \bar{c}(z)\,c(w) \right\rangle \\ & + PV \Big\} \end{aligned} \tag{44}$$

The first term is

$$(2\pi)^2 \left(\frac{\tilde{m}^2}{4} \right) m^2 \int \frac{d^2p}{(2\pi)^2}\, e^{-ip\cdot x} \int \frac{d^2k}{(2\pi)^2} \frac{1}{[k^2 + \tilde{m}m]} \frac{1}{[(p-k)^2 + \tilde{m}m]} + PV,$$

which was calculated in [1] to give

$$\frac{2\pi}{12}\, \partial\bar{\partial}\, \delta^2(z-w) \tag{45}$$

in the limit $m, \tilde{m} \to 0$ and $M_\rho, \tilde{M}_v \to \infty$, given that appropriate Pauli-Villars relations on the masses are satisfied. Similarly, the third term is

$$(2\pi)^2\, m^2\, \tilde{m}^2 \int \frac{d^2p}{(2\pi)^2}\, e^{-ip\cdot x} \int \frac{d^2k}{(2\pi)^2} \frac{1}{[k^2 + \tilde{m}m]} \frac{1}{[(p-k)^2 + \tilde{m}m]} + PV,$$

which gives

$$\frac{8\pi}{12}\, \partial\bar{\partial}\, \delta^2(z-w) \tag{46}$$

in the limit. The second term can be written, with the notation $k \equiv k_1 + ik_2$

in numerators, as

$$
\begin{aligned}
&(2\pi)^2\, m\tilde{m} \int \frac{d^2p}{(2\pi)^2}\, e^{-ip\cdot x} \int \frac{d^2k}{(2\pi)^2}\, \frac{\bar{k}}{[k^2+\tilde{m}m]}\, \frac{p-k}{[(p-k)^2+\tilde{m}m]} + PV \\
&\quad = (2\pi)^2\, m\tilde{m}\, \frac{1}{2} \int \frac{d^2p}{(2\pi)^2}\, e^{-ip\cdot x} \int \frac{d^2k}{(2\pi)^2}\, \frac{\bar{k}(p-k)+k(\bar{p}-\bar{k})}{[k^2+\tilde{m}m]\,[(p-k)^2+\tilde{m}m]} + PV \\
&\quad = (2\pi)^2\, m\tilde{m}\, \frac{1}{2} \int \frac{d^2p}{(2\pi)^2}\, e^{-ip\cdot x} \int \frac{d^2k}{(2\pi)^2} \left\{ -\frac{1}{(p-k)^2+\tilde{m}m} - \frac{1}{k^2+\tilde{m}m} \right. \\
&\qquad\qquad \left. + \frac{p^2+2\,\tilde{m}m}{[k^2+\tilde{m}m]\,[(p-k)^2+\tilde{m}m]} + PV \right\}.
\end{aligned}
$$

The first two terms are proportional to

$$
\tilde{m}m\, \ln \frac{\Lambda^2}{\tilde{m}m} + PV
$$

which vanishes by the usual conditions on the Pauli-Villars masses and statistics. The Fourier transform of the last term may be rewritten as [1]

$$
(2\pi)^2\, \frac{1}{2} \left(\frac{\pi}{3}\right) \frac{1}{2\,(2\pi)^2} \left\{ \int_{2m}^{\infty} d\mu\, c(\mu,m)\, \frac{\mu^4}{p^2+\mu^2} \left[\frac{p^2}{m^2}+2\right] + PV \right\},
$$

where the spectral function $c(\mu,m)$ is given in [1]. To save space on notation, we have set $\tilde{m}=m$. The contribution proportional to the $+2$ in the square brackets was calculated in [1] to give, in conjunction with the Pauli-Villars contributions, the result

$$
\frac{\pi}{6}\, p^2. \tag{47}
$$

The term proportional to p^2/m^2 may be rewritten as

$$
\begin{aligned}
&(2\pi)^2\, \frac{1}{2} \left(\frac{\pi}{3}\right) \frac{1}{2\,(2\pi)^2}\, p^2 \left\{ \int_{2m}^{\infty} d\mu\, c(\mu,m)\, \frac{\mu^4}{p^2+\mu^2}\, \frac{1}{m^2} \left[\mu^2 - \frac{\mu^2 p^2}{p^2+\mu^2}\right] + PV \right\} \\
&\quad = \frac{1}{4} \left(\frac{\pi}{3}\right) p^2 \int_1^{\infty} \frac{3}{2}\, \frac{d\eta}{\eta^4}\, \frac{1}{\sqrt{\eta^2-1}} \left[4\eta^2 - \frac{4p^2\eta^2}{p^2+2m^2\eta^2} + PV\right].
\end{aligned}
$$

The first term is independent of m, and will vanish in conjunction with the Pauli-Villars contributions by the usual condition $\sum c_i = 0$ on the statistics.

The second term vanishes for the Pauli-Villars fields in the limit $M \to \infty$, and only the ghost contribution remains, which for $m \to 0$ is given by

$$\begin{aligned} \frac{1}{4}\left(\frac{\pi}{3}\right) p^2 \left(\frac{3}{2}\right) \int_1^\infty \frac{d\eta}{\eta^4} \frac{1}{\sqrt{\eta^2-1}} (-4\eta^2) &= \frac{1}{4}\left(\frac{\pi}{3}\right) p^2 (-4) \left(\frac{3}{2}\right) \\ &= -\frac{\pi}{2} p^2. \end{aligned} \tag{48}$$

Adding (47) and (48), and taking the Fourier transform, we obtain the result

$$\frac{16\pi}{12} \partial\bar{\partial}\, \delta^2(z-w) \tag{49}$$

for the second term in (44). Adding the three contributions (45), (46) and (49), we finally obtain, for the ghosts and their partners

$$\langle T_{z\bar{z}} T_{w\bar{w}} \rangle = \frac{26\pi}{12} \partial\bar{\partial}\, \delta^2(z-w).$$

From this, we can read off [1] the ghost contribution

$$c = -26$$

to the anomaly.

14 Conclusion

In this article we discussed a covariant functional integral approach to the quantization of the bosonic string. We showed that interesting operators could be renormalized as true tensors, independently of whether the theory has a Weyl anomaly. As a result, issues related to the anomaly could be isolated more clearly. This method of operator renormalization is in principle of wider applicability to covariant theories that are not Weyl invariant.

Also of wider applicability in generally covariant theories is our construction of a class of background-independent path integral measures, as well as the construction of the BRST action from first principles and the discussion of its background invariance. Interestingly, the BRST action could not be written in a background-independent way, although we showed that one has some freedom in shifting the dependence on a background metric from one set of terms to another. However, it should be emphasized that this background dependence is completely innocuous, since no dependence on the background metric remains in the physical results.

Overall, the familiar string theory results are all reproduced in the current formalism. What is interesting, and instructive, is that they are encoded in the formalism somewhat differently from the usual approaches. The formalism of this paper separates the issue of operator renormalization very cleanly from the concern of anomaly analysis.

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